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Recent Advances in DualSPHysics: ALE-SPH formulation, non-conservative pressure, and divergence cleaning

Renato Vacondio

Standard DualSPHysics formulation

Weakly compressible form of Navier – Stokes equations

$$\frac{\mathrm{d}\,\rho}{\mathrm{d}\,t} = -\rho \nabla \cdot \mathbf{u}$$

Continuity Equation

$$\frac{\mathrm{d}\ldots}{\mathrm{d}\,t} :=$$ Lagrangian/Material derivative

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{dt}} = -\frac{1}{\rho}\nabla P + \Gamma + \mathbf{f}$$

Conservation of Momentum

Equation of state: $$P = \frac{c_o^2 \rho_w}{\gamma}\left(\left(\frac{\rho}{\rho_w}\right)^{\gamma} - 1\right)$$

- the artificial speed of sound c_0 is chosen such Ma < 0.1,

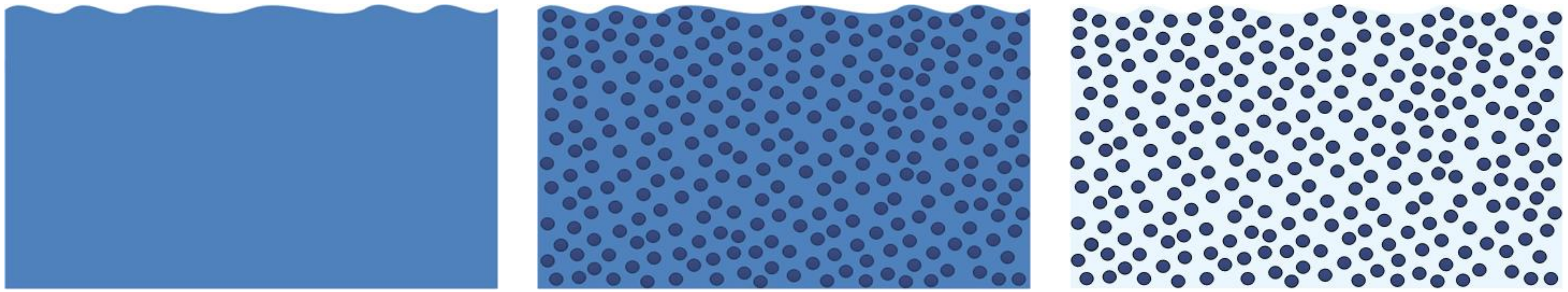
- density variations remain within 1%

where:

- $\mathbf{u}$ fluid velocity vector
- t time
- ρ fluid density
- P pressure
- Γ viscous term (SPH diffusion / dissipation)
- $\boldsymbol{f}$ body forces (e.g. gravity)

The shortest possible introduction to SPH

Continuous is represented using particles which move according to the governing dynamics.



Continuous fluid

Set of particles

SPH is based on integral interpolants

The fundamental principle is to approximate any function *f*(**r**) by

$$f(\mathbf{r}) = \int_{\Omega} f(\mathbf{r}')\, W(\mathbf{r} - \mathbf{r}', h)\, d\mathbf{r}'$$

$$f_i(\mathbf{r}) \approx \sum_{j=1}^{N} f(\mathbf{r}_j)\, W(\mathbf{r}_i - \mathbf{r}_j, h)\, V_j$$

Continuous level

Discrete level

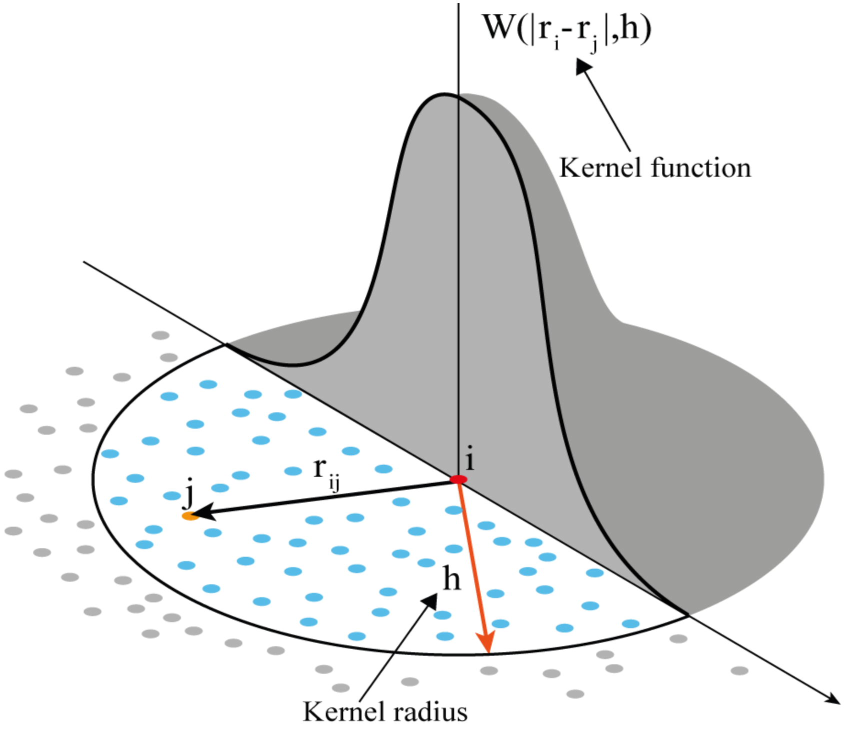
The shortest possible introduction to SPH

SPH spatial interpolation can be used for differential operators:

$$\langle \nabla f(\mathbf{r}) \rangle = \int_{\Omega} f(\mathbf{r}')\, \nabla W(\mathbf{r}-\mathbf{r}', h)\, \mathrm{d}\,\mathbf{r}'$$

$$\Big\downarrow$$

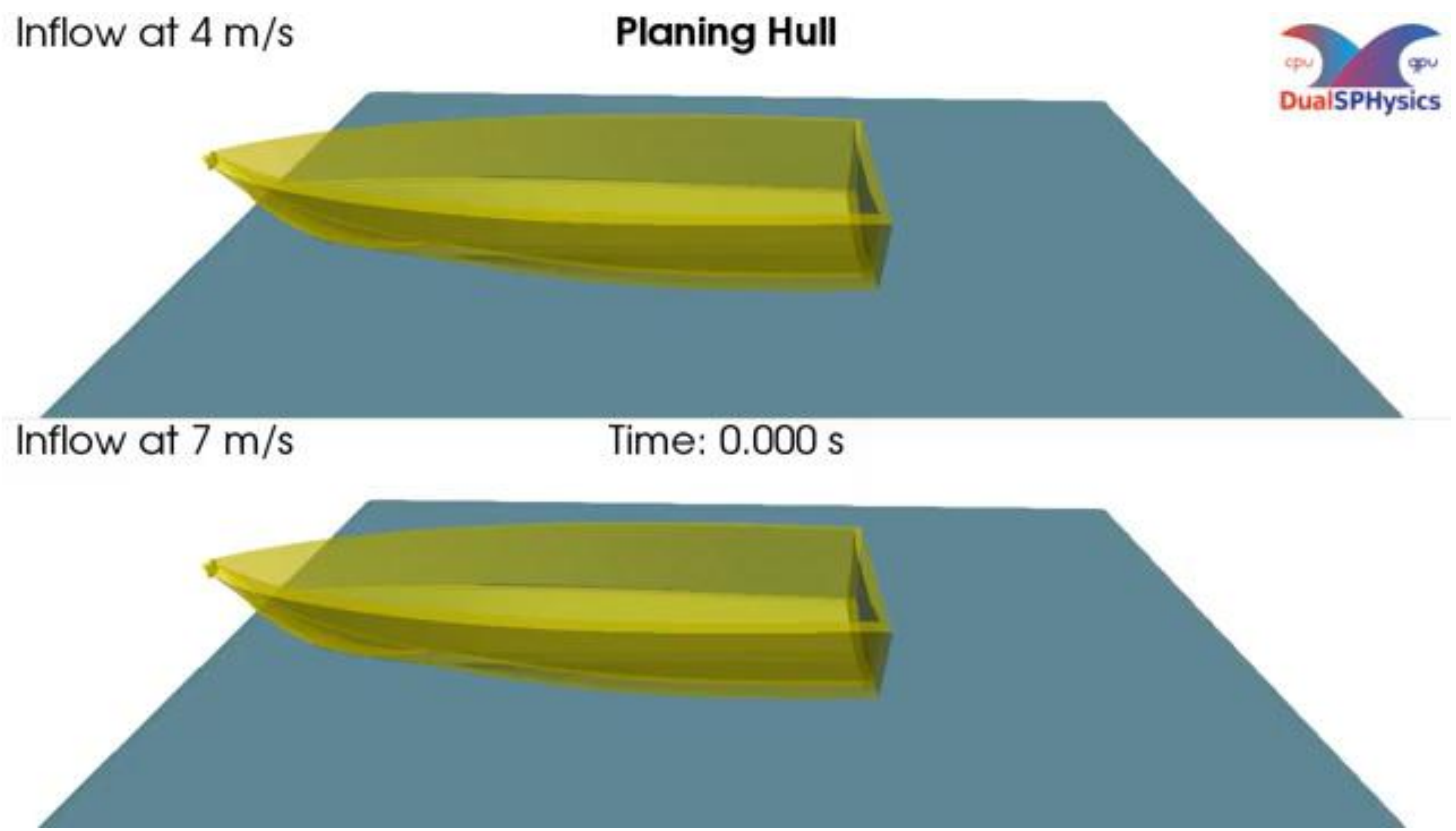
$$\langle \nabla f(\mathbf{r}_i) \rangle \approx \sum_{j=1}^{N} f(\mathbf{r}_j)\, \nabla W(\mathbf{r}_i-\mathbf{r}_j, h)\, V_j$$



W is the so-called SPH kernel

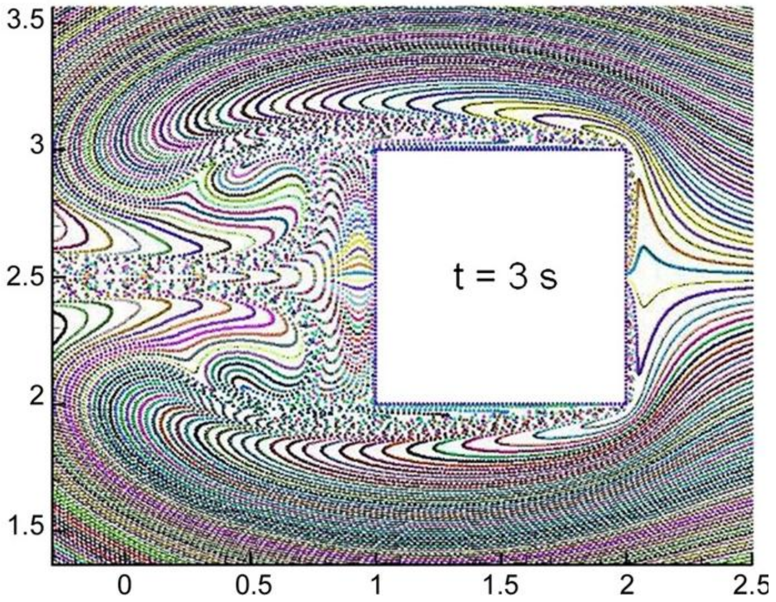
h is the smoothing length that controls the size of the Kernel (typically 2h)

Lagrangian + SPH is amazing!!!

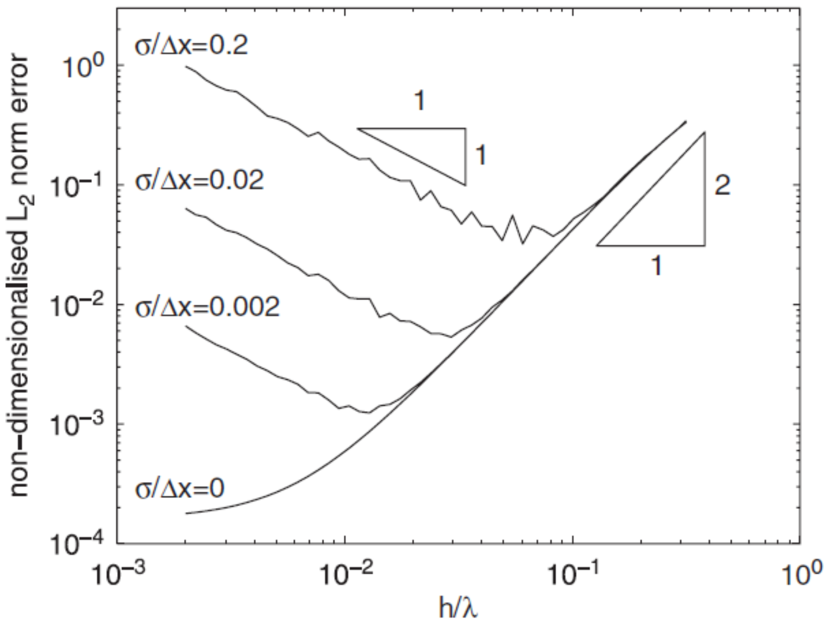


Is it really?

- In fully Lagrangian schemes, there is no control on the particle distribution
- The accuracy of SPH spatial operators depends on the quality of the particle distribution



Snapshots of the purely Lagrangian evolution of the **flow past a square shaped obstacle** (Oger et al. JCP 2016)



Convergence analysis of (Quinan et al. IJNME 2006)

SPH discretization

$$\frac{\mathrm{d}\,\rho}{\mathrm{d}\,t} = -\rho \nabla \cdot \mathbf{u}$$

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{dt}} = -\frac{1}{\rho} \nabla P + \Gamma + \mathbf{f}$$

$\frac{\mathrm{d}}{\mathrm{d}\,t}$: Lagrangian derivative

$$\frac{\mathrm{d}\,\rho_a}{\mathrm{d}\,t} = -\rho_a \sum_b (\mathbf{u}_b - \mathbf{u}_a) \cdot \nabla W_{ab} V_{\mathrm{b}}$$

$$\frac{\mathrm{d}\,\mathbf{u}_a}{\mathrm{d}\,t} = -\frac{1}{\rho_a} \sum_b (P_b + P_{\mathrm{a}}) \nabla_a W_{ab} V_{\mathrm{b}} + \Gamma + \mathbf{f}$$

$$\frac{\mathrm{d}\,\mathbf{x}_a}{\mathrm{d}\,t} = \mathbf{u}_a + \frac{\delta \mathbf{x}_a}{\mathrm{d}\,t}$$

Shifting velocity: not fully Lagrangian

where $\delta \mathbf{x}_a$ is the shifting

How do we shift?

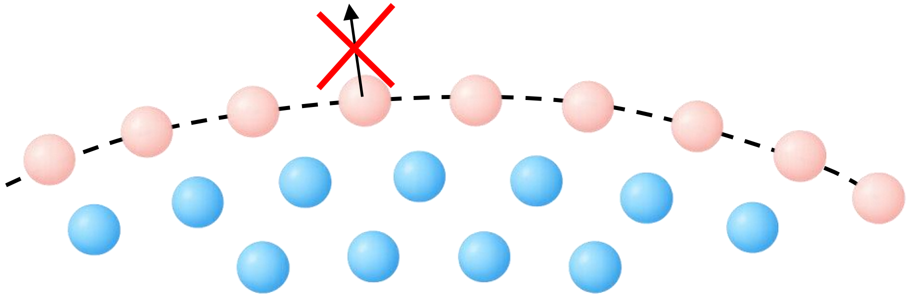
$$\delta \mathbf{x}_a = -D\nabla C_a \qquad \nabla C_a = \sum_b 1\nabla W_{ab} V_b$$

Particle concentration

Skillen et al. (2013) CMAME $D = A\|\boldsymbol{u}_a\|hdt$ → Standard shifting for DualSPHysics (shifting<4)

Lind et al. (2012) JCP $D = Ah^2$ → Advanced shifting (shifting=4)

Main issue: on the free surface the kinematic conditions requires that particles cannot be shifted in the normal direction.



Advanced Shifting
(better particle distribution)

```xml
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<parameter key="ShiftAdvCoef" value="-0.01" comment="Coefficient for advanced shifting computation (default=-0.01)" />
<parameter key="ShiftTFS" value="0" comment="Threshold to detect free surface. Typically 1.5 for 2D and 2.75 for 3D (default=0)" />
```

(Ricci et al. CPM 2025)

How free surface particles are identified

Criterion 1 - divergence of the position

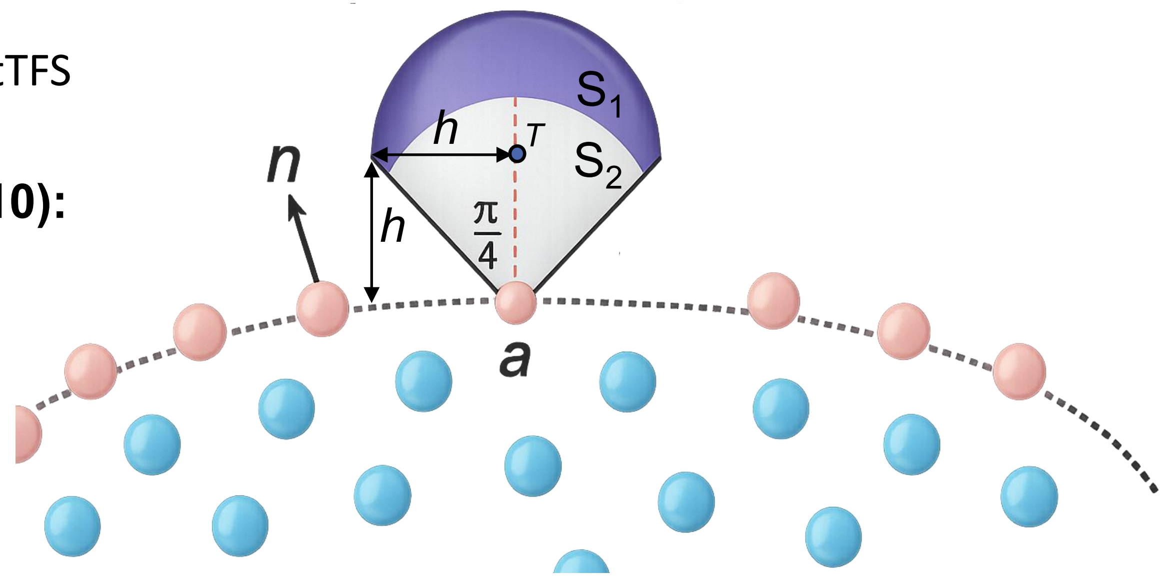
$$\nabla \cdot \mathbf{x}_a = \sum_{\mathrm{b}} (\mathbf{x}_a - \mathbf{x}_{\mathrm{b}}) \nabla W_{ab} V_b$$

$$\nabla \cdot \boldsymbol{x}_a < A_{tfs} \qquad A_{tfs} = \begin{cases} 1.5 - 1.7 \text{ in 2D} \\ 2.75 \text{ in 3D} \end{cases}$$

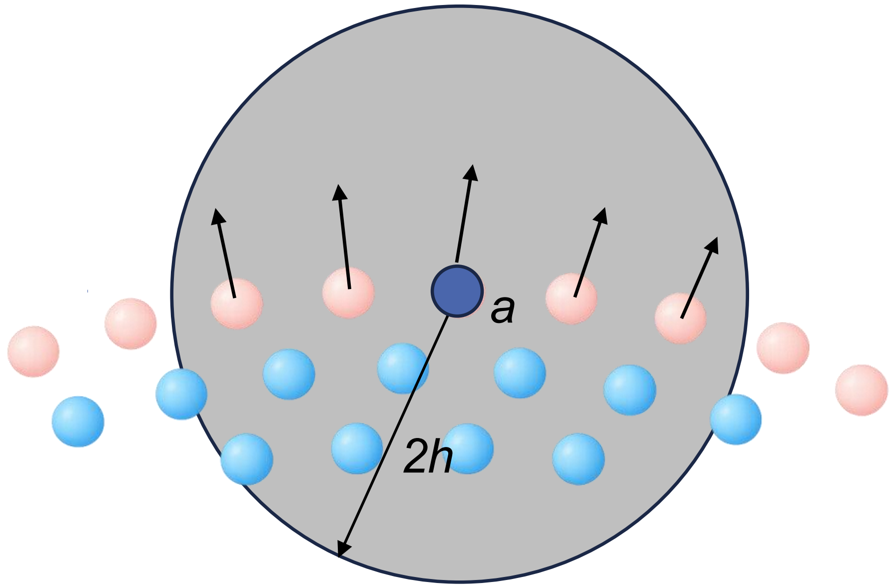
ShiftTFS

Criterion 2- umbrella region (Marrone et al. JCP 2010):

$$\begin{cases} \forall b \in \mathbb{N}\left[|\boldsymbol{x}_{ba}| \geqslant \sqrt{2}h, |\boldsymbol{x}_{bT}| < h\right] & \Rightarrow \quad a \notin \mathbb{F} \\ \forall b \in \mathbb{N}\left[|\boldsymbol{x}_{ba}| < \sqrt{2}h, |\boldsymbol{n} \cdot \boldsymbol{x}_{bT}| + |\boldsymbol{\tau} \cdot \boldsymbol{x}_{bT}| < h\right] & \Rightarrow \quad a \notin \mathbb{F} \\ otherwise & \quad a \in \mathbb{F} \end{cases}$$



Shifting close to the free surface



Normals are computed using the following SPH equation:

$$\mathbf{n}_a = -\frac{\mathbf{L}_a \cdot \nabla C_a}{|\mathbf{L}_a \cdot \nabla C_a|}$$

$\mathbf{L}_a$ is Bonet & Lok correction matrix

$\mathbf{n}_a$ are then smoothed using **only the free – surface particles**:

$$\bar{\mathbf{n}}_a = \frac{\sum_{b\in\Omega_F} \mathbf{n}_b W_{ab} V_b}{\sum_{b\in\Omega_F} W_{ab} V_b}$$

For free-surface particles shifting is applied only in the tangential direction:

$$\delta\mathbf{x}_{\tau a} = \delta\mathbf{x}_a \cdot (\mathbf{I} - \bar{\mathbf{n}}_a \otimes \bar{\mathbf{n}}_a)$$

For particles near the free surface the shifting in normal direction is reduced proportionally to the distance to the free surface

ALE-SPH formulation
(how we fix the inconsistency introduced by shifting)

```
<parameter key="ShiftAdvALE" value="1" comment="Shifting ALE corrective terms 0:Inactive 1:Active (default=0)" />
```

(Ricci et al. CPM 2025)

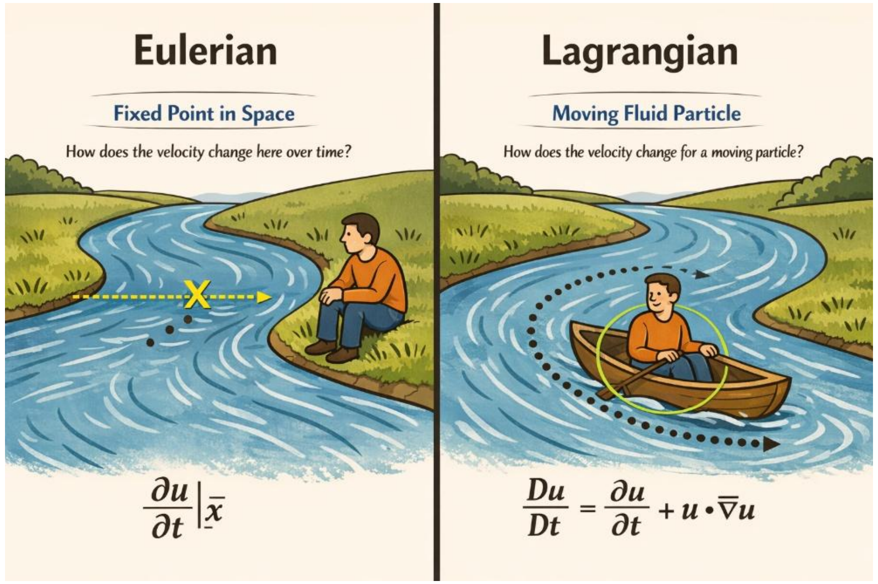
From the Reynolds transport theorem:

$\phi(\mathbf{x}, \mathrm{t})$ is a generic scalar variable

$$\frac{\mathrm{d}\,\phi(\mathbf{x},\mathrm{t})}{\mathrm{d}\,t} = \frac{\partial\,\phi(\mathbf{x},\mathrm{t})}{\partial\,t} + \mathbf{u}\cdot\nabla\phi(\mathbf{x},\mathrm{t})$$

Eulerian derivative

Material/Lagrangian derivative



In an Arbitrary Lagrangian-Eulerian (ALE) framework this becomes:

$$\frac{\mathrm{d}\,\phi(\mathbf{x},\mathrm{t})}{\mathrm{d}\,t} = \left.\frac{\partial\,\phi(\mathbf{x},\mathrm{t})}{\partial\,t}\right._{\mathrm{ALE}} + (\mathbf{u}-\mathbf{u}_0)\cdot\nabla\phi(\mathbf{x},\mathrm{t})$$

$\mathbf{u}_0$ is the transport velocity

ALE formulation for SPH

$$\frac{\mathrm{d}\,\mathbf{x}}{\mathrm{d}\,t} = \mathbf{u} + \frac{\delta\mathbf{x}}{\mathrm{d}\,t} \Rightarrow \mathbf{u}_0 = \mathbf{u} + \frac{\delta\mathbf{x}}{\mathrm{d}\,t} \Rightarrow \hat{\mathbf{u}} = \frac{\delta\mathbf{x}}{\mathrm{d}\,t} = (\mathbf{u}_0 - \mathbf{u})$$

$\mathbf{u}_0$ → Transport velocity

$\hat{\mathbf{u}}$ → Shifting velocity

Substituting the ALE derivative in the NS equations we obtain:

$$\begin{cases} \dfrac{\mathrm{d}\,\rho}{\mathrm{d}\,t} = -\rho\nabla\cdot\mathbf{u} \\ \dfrac{\mathrm{d}\mathbf{u}}{\mathrm{dt}} = -\dfrac{1}{\rho}\nabla P + \Gamma + \mathbf{f} \end{cases} \Rightarrow \begin{cases} \dfrac{\partial\,\rho}{\partial\,t}_{\mathrm{ALE}} - \hat{\mathbf{u}}\cdot\nabla\rho = -\rho\nabla\cdot\mathbf{u} \\ \dfrac{\partial\,\mathbf{u}}{\partial\,t}_{\mathrm{ALE}} - \hat{\mathbf{u}}\cdot\nabla\mathbf{u} = -\dfrac{1}{\rho}\nabla P + \Gamma + \mathbf{f} \end{cases}$$

The advection terms need to be computed if particles are shifted

ALE formulation for SPH – Continuity Equation

Continuity equation:

$$\frac{\partial \rho}{\partial t}_{\mathrm{ALE}} - \hat{\mathbf{u}} \cdot \nabla\rho = -\rho\nabla \cdot \mathbf{u} \quad \Rightarrow \quad \frac{\partial \rho}{\partial t}_{\mathrm{ALE}} = -\rho\nabla \cdot (\mathbf{u} + \hat{\mathbf{u}}) + \nabla \cdot (\rho\hat{\mathbf{u}})$$

(1) (2)

(1) $\Rightarrow$ $\rho_{\mathrm{a}}\nabla \cdot (\mathbf{u} + \hat{\mathbf{u}})_{\mathrm{a}} = \rho_{\mathrm{a}} \sum_{\mathrm{b}}(\mathbf{u}_{\mathrm{b}} - \mathbf{u}_{\mathrm{a}})\nabla W_{ab} V_b + \rho_{\mathrm{a}} \sum_{\mathrm{b}}(\hat{\mathbf{u}}_{\mathrm{b}} - \hat{\mathbf{u}}_{\mathrm{a}})\nabla W_{ab} V_b$

This is consistent with the standard SPH operator.

(2) $\Rightarrow$ $\nabla \cdot \left(\widehat{\rho\mathbf{u}}\right)_{\mathrm{a}} = \sum_{\mathrm{b}}(\rho_{\mathrm{b}}\hat{\mathbf{u}}_{\mathrm{b}} + \rho_{\mathrm{a}}\hat{\mathbf{u}}_{\mathrm{a}})\nabla W_{ab} V_b$

Symmetric and thus it conserves mass

(Antuono et al. C&F 2021)

ALE formulation for SPH – Momentum equation

Momentum equation:

$$\frac{\partial \mathbf{u}}{\partial t}_{\mathrm{ALE}} - \hat{\mathbf{u}} \cdot \nabla \mathbf{u} = \ldots \Rightarrow \frac{\partial \mathbf{u}}{\partial t}_{\mathrm{ALE}} = +\underbrace{\nabla \cdot (\mathbf{u} \otimes \hat{\mathbf{u}})}_{(1)} - \underbrace{\mathbf{u}\nabla \cdot \hat{\mathbf{u}}}_{(2)} + \ldots$$

(1) $\Rightarrow$ $\nabla \cdot (\mathbf{u} \otimes \hat{\mathbf{u}})_{\mathrm{a}} = \sum_{\mathrm{b}} (\mathbf{u}_{\mathrm{a}} \otimes \hat{\mathbf{u}}_{\mathrm{a}} + \mathbf{u}_{\mathrm{b}} \otimes \hat{\mathbf{u}}_{\mathrm{b}}) \nabla W_{ab} V_b$

Symmetric and thus it conserves linear momentum (not angular momentum)

(2) $\Rightarrow$ $\nabla \cdot \hat{\mathbf{u}}_{\mathrm{a}} = \sum_{\mathrm{b}} (\hat{\mathbf{u}}_{\mathrm{b}} - \hat{\mathbf{u}}_{\mathrm{a}}) \nabla W_{ab} V_b$

consistent with the continuity equation (usually negligible).

(Antuono et al. C&F 2021)

No more Maths…

Propagation of regular waves

Motivation:

- Check three configurations: Standard shifting, Advanced Shifting, Advanced Shifting + ALE

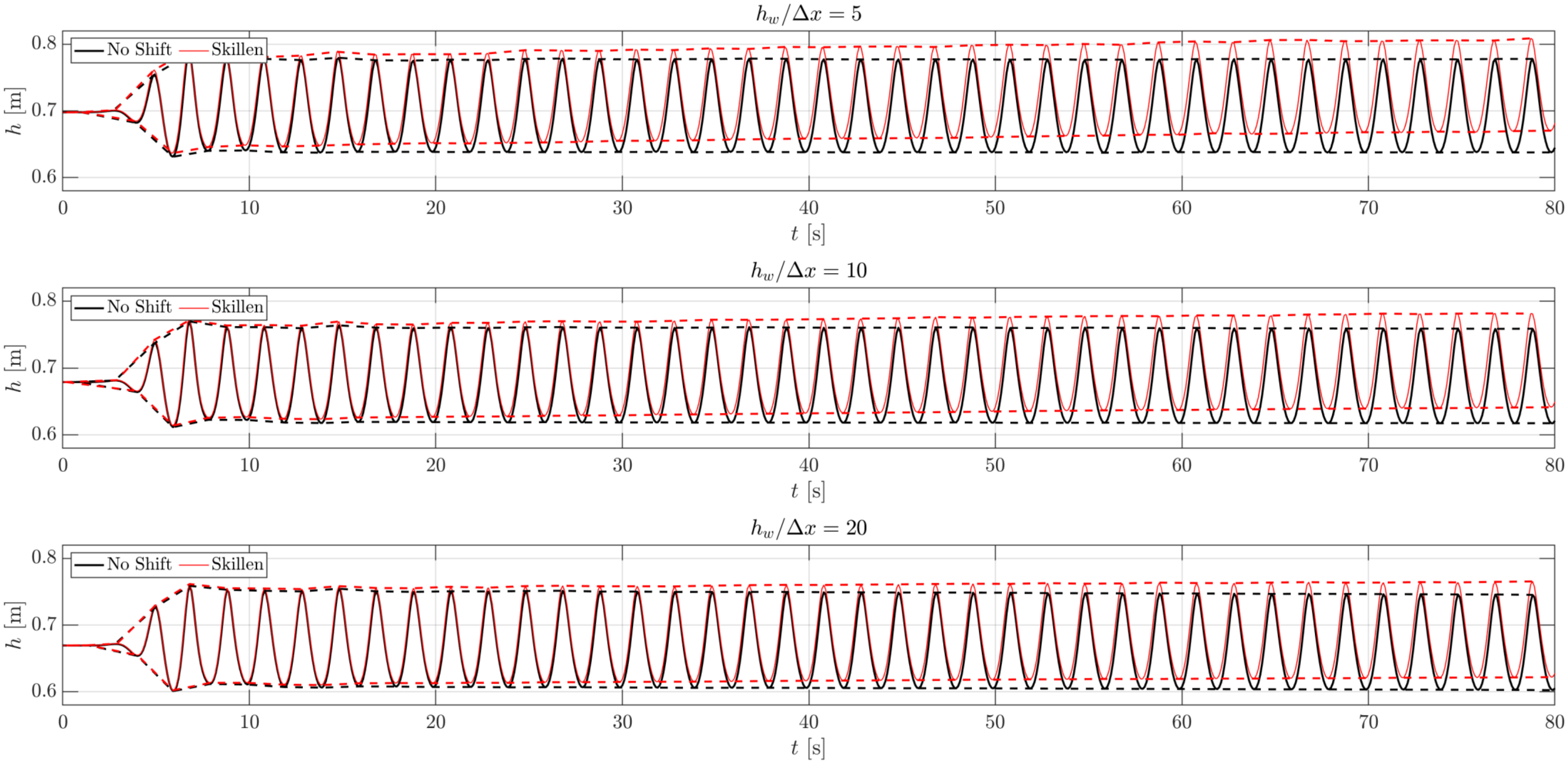
Numerical setup:

- **Same setup of main/09_PistonWaves**
- H/Dp=2.0
- Artificial viscosity = 0.01
- Skillen coefficient = -2
- Advanced Shifting coefficient = -0.01

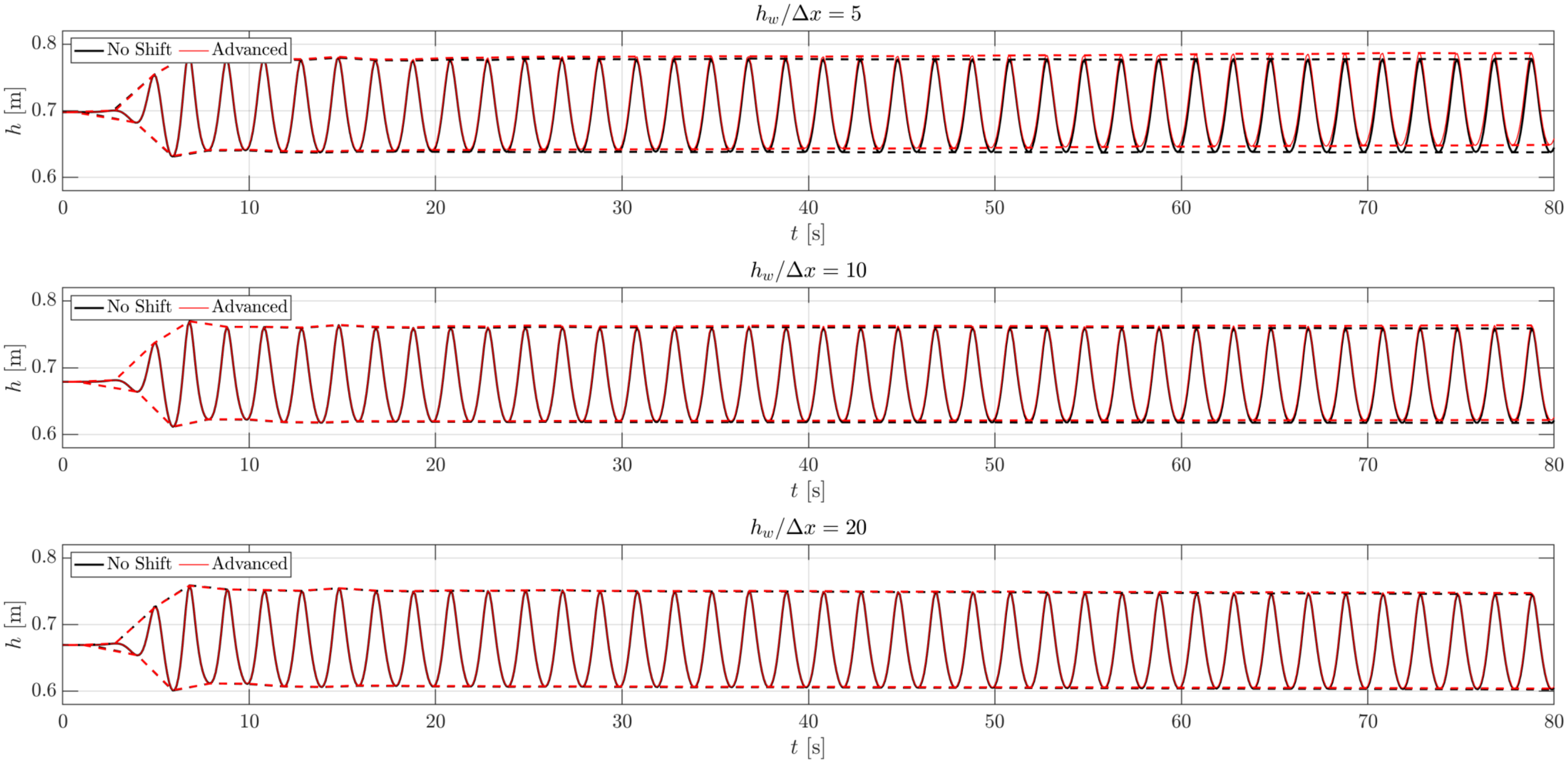
For each formulation:

- **Dp=0.03,0.015,0.075**, corresponding to 5,10,20 particles per wave height
- Monitor: Wave elevation at x=6.5m

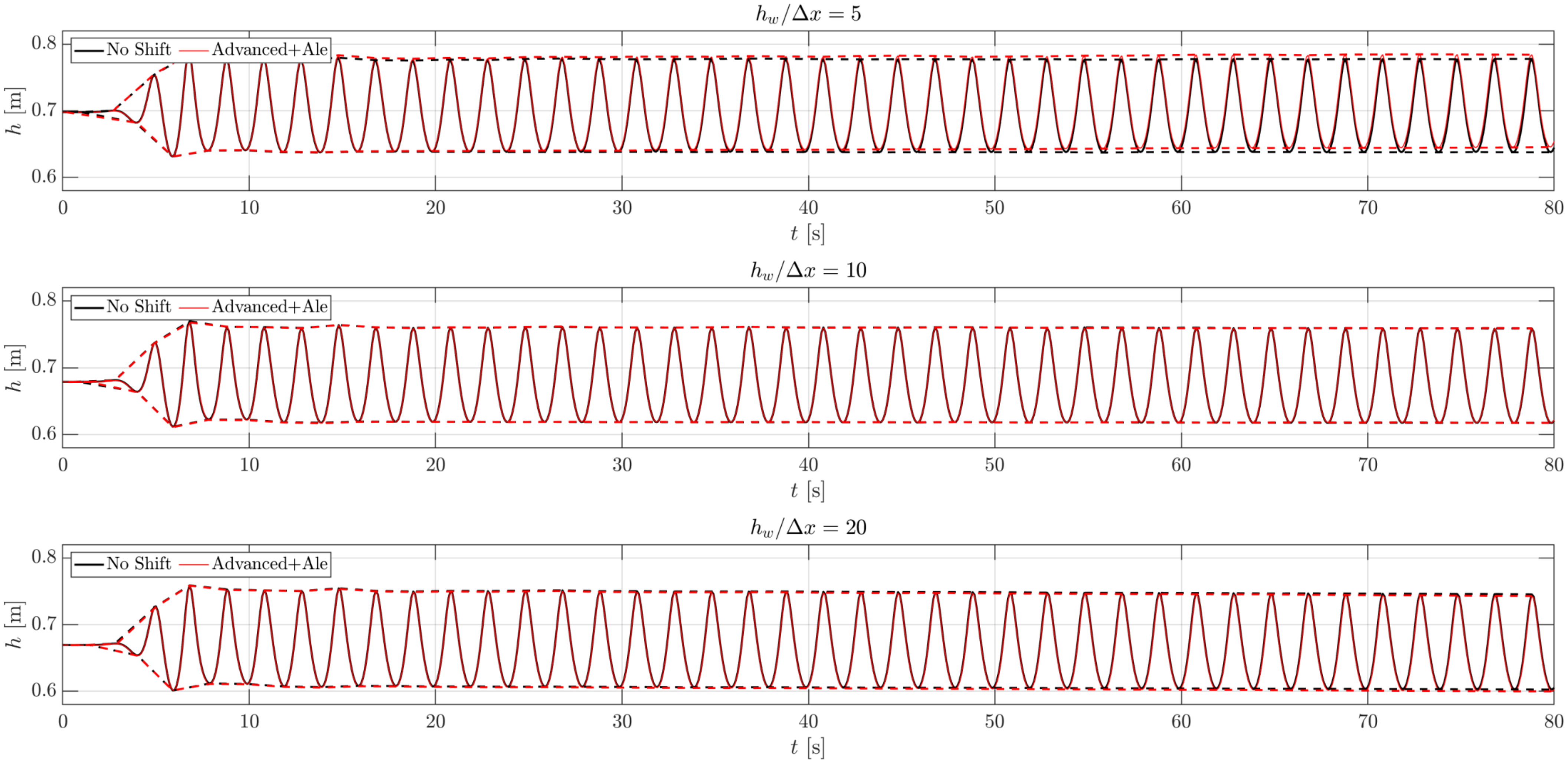
Propagation of regular waves: Standard shifting (Skillen)



Propagation of regular waves: Advanced shifting



Propagation of regular waves: Advanced shifting + ALE



Advanced shifting + ALE: conclusions

Conclusions:

- Advanced shifting + ALE performs better in terms of **volume conservation** with respect to Skillen

- Advanced shifting + ALE provide a **better particles arrangement** near the free-surface.

Non-conservative pressure formulation (how we improve consistency)

```
<parameter key="ShiftAdvNCPress" value="1" comment="Non-Conservative pressure formulation 0:Inactive 1:Active (default=0)" />
```

Non Conservative pressure

$$\frac{\mathrm{d}\,\mathbf{u}_a}{\mathrm{d}\,t} = -\frac{1}{\rho_a}\sum_b f(P_\mathrm{a}, P_\mathrm{b})\,\nabla_a W_{ab} V_\mathrm{b} + \ldots$$

$$f(P_a, P_b) = \begin{cases} \sum_b (P_b + P_a)\ \nabla_a W_{ab}\, V_b & a \in \Omega_F \cup\in \Omega_{NF} \\ \sum_b (P_b - P_a)\ L_a \nabla_a W_{ab}\, V_b & a \in \Omega_I \end{cases}$$

Particles near the free surface

Internal particles

$\mathbf{L}_a$ is Bonet & Lok correction matrix

The formulation is non conservative

First order consistent inside the fluid

This is fundamental if **Variable Resolution is used**

(Michel et al. PoF 2023)

Propagation of regular waves: NC

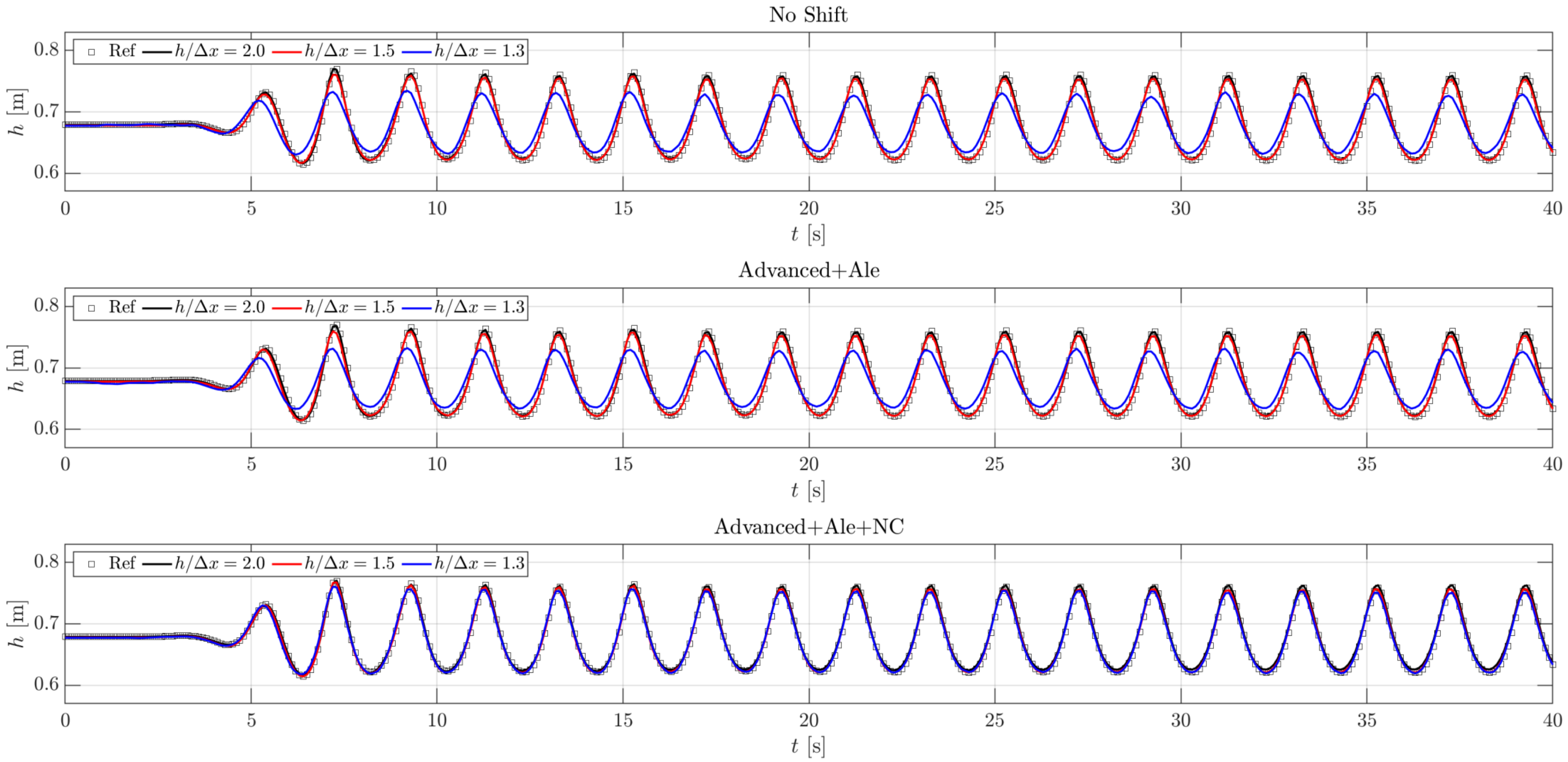
Motivation:

- Demonstrate the better convergence property of the non-conservative formulation for propagation of waves

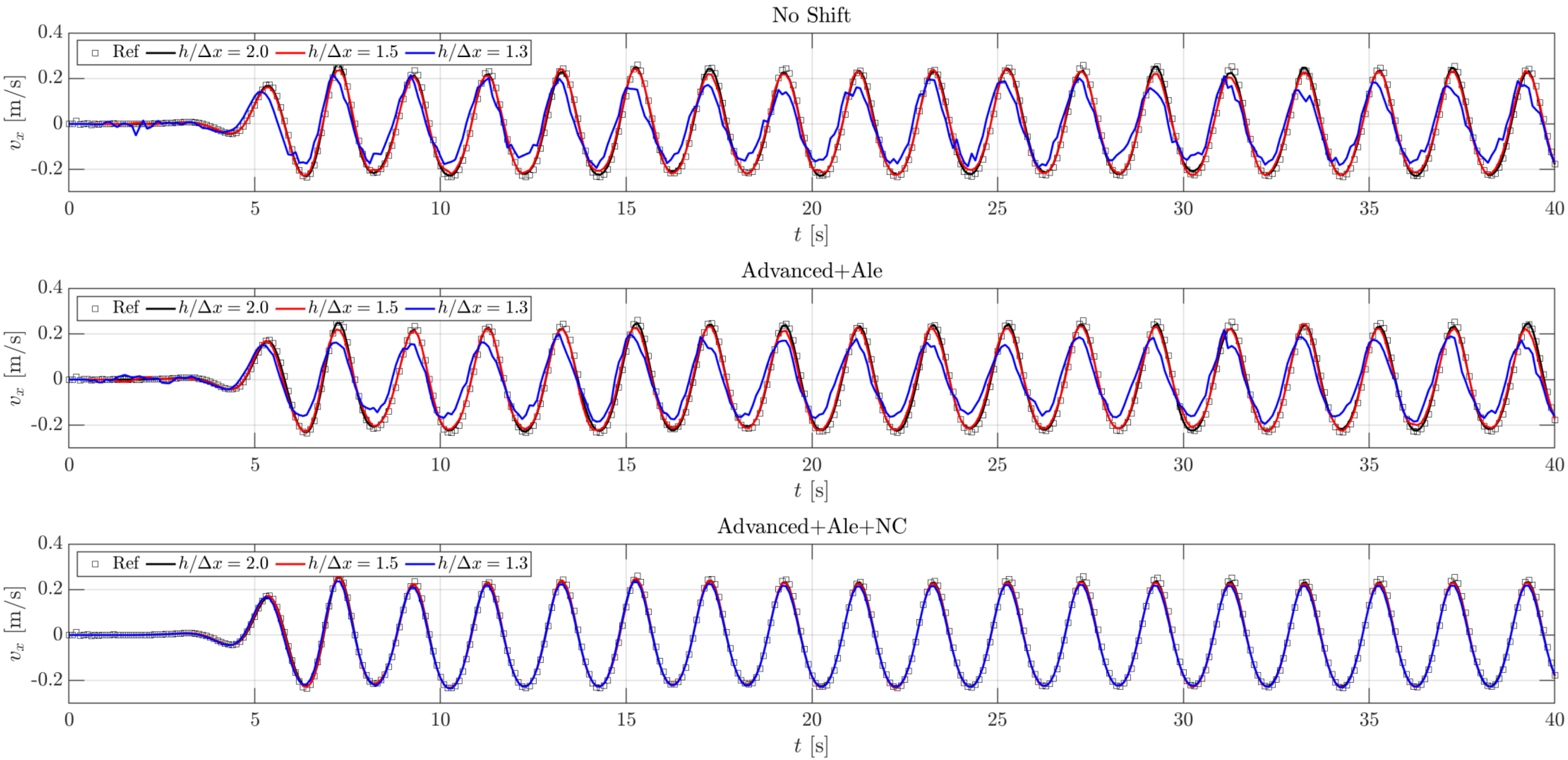
Numerical setup:

- Same setup of main/09_PistonWaves
- **h/Dp = 2.0,1.5,1.3**
- Artificial viscosity = 0.01
- Advanced Shifting coefficient = -0.01
- **Dp=0.015, corresponding to 10 particles per wave height**

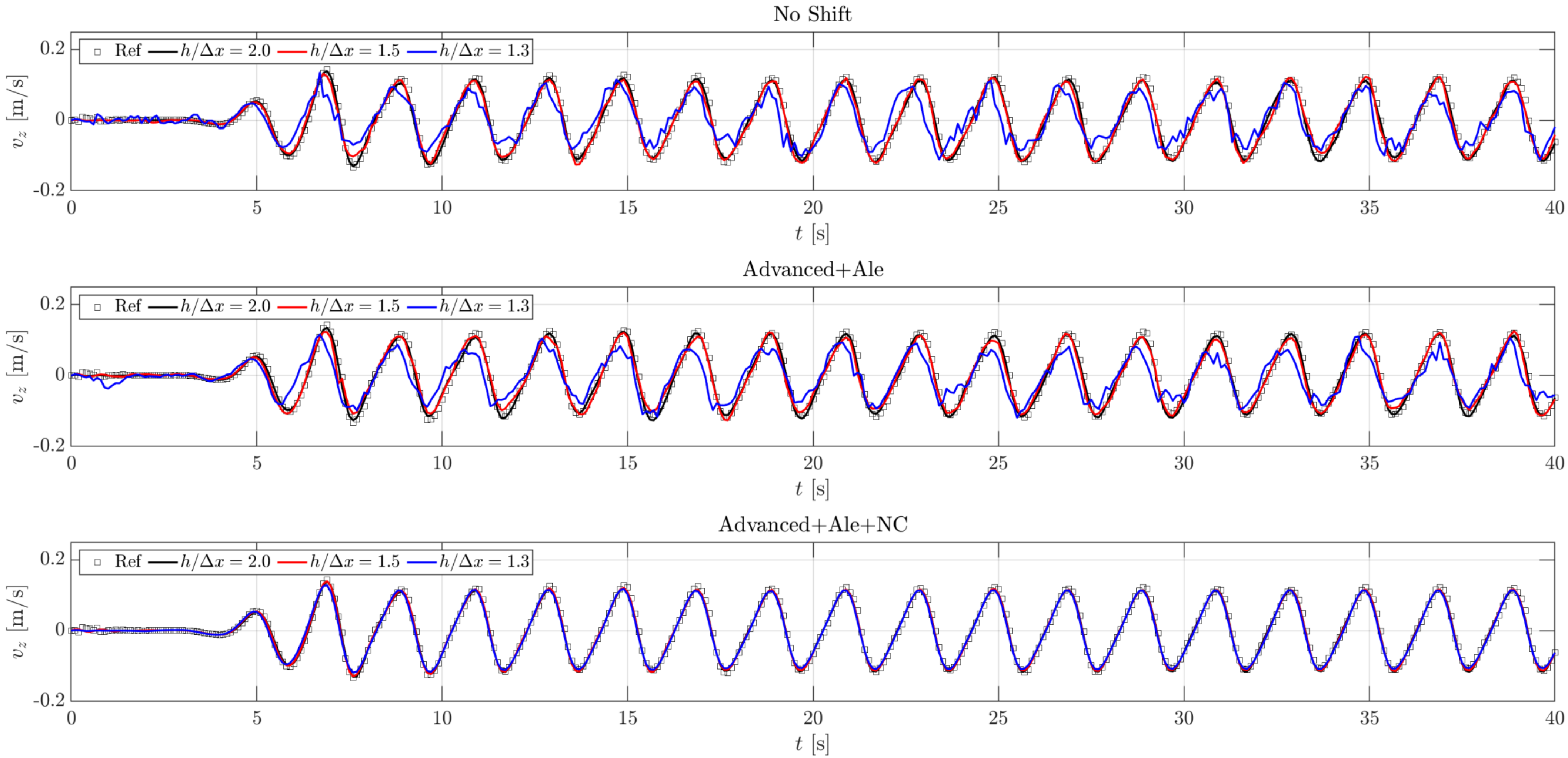
Propagation of regular waves: Non-Conservative *h*



Propagation of regular waves: Non-Conservative v_x



Propagation of regular waves: Non-Conservative v_z



Conclusions on non-conservative pressure :

- It **reduces the numerical noise.**
- It **cannot be adopted without Advanced Shifting + ALE**.
- It simulates regular waves with h/Dp<1.5
- Important for 3D simulation: low h/Dp values translate in **faster simulation**

Bonus Maths…

Divergence cleaning

```
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```

Formulation considered: WC-SPH

Classical WCSPH (Monaghan 1994)

$$\frac{d\rho_a}{dt} = \rho_a \sum_b \mathbf{u}_{ab} \cdot \nabla W_{ab} V_b + h c_0 D_a$$

$$\frac{d\mathbf{u}_a}{dt} = -\frac{1}{\rho_a} \sum_b (P_a + P_b) \nabla W_{ab} V_b + \Gamma$$

- D_a is the density diffusion term
- Viscous term (Lo & Shao 2002)

$$\Gamma = \langle v_a \nabla^2 \mathbf{u}_a \rangle = \sum_b m_b \frac{(\mu_a + \mu_b)}{\rho_a \rho_b} \frac{\mathbf{x}_{ab}}{x_{ab}^2} \cdot \nabla W_{ab} \mathbf{u}_{ab}$$

- Artificial viscosity (Monaghan 1992):

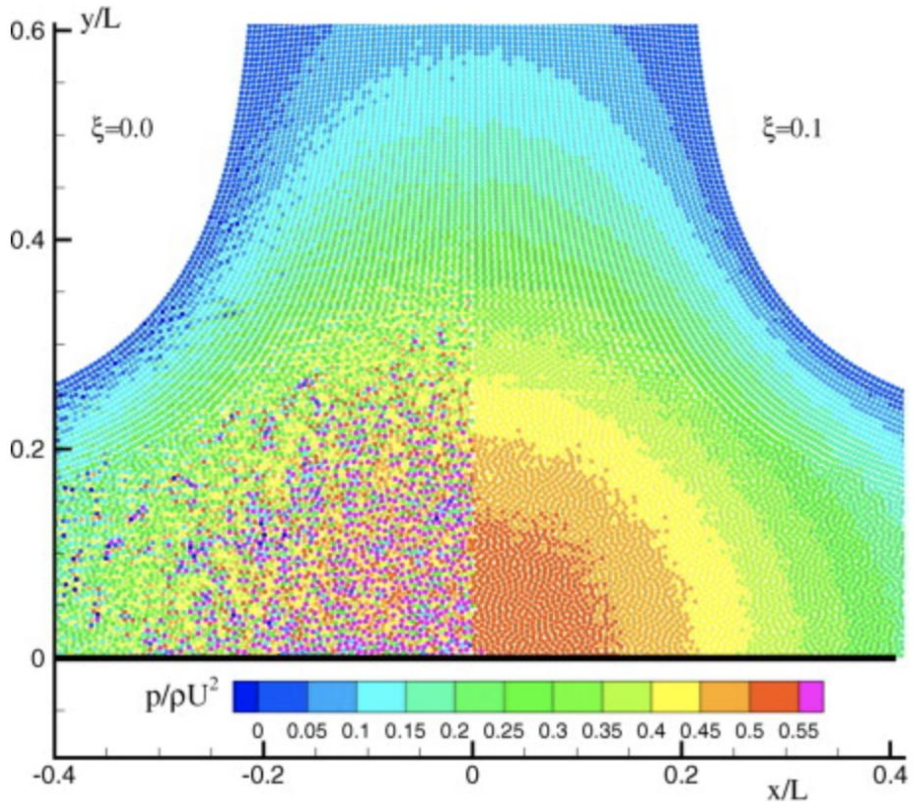
$$\Gamma = \sum_b m_b \Pi_{ab} \nabla W_{ab} \mathbf{u}_{ab}$$

Density Diffusion Term

$$D_a = \sum_b \psi_{ab} \nabla W_{ab} V_b$$

ψ_{ab} depends on the adopted formulation

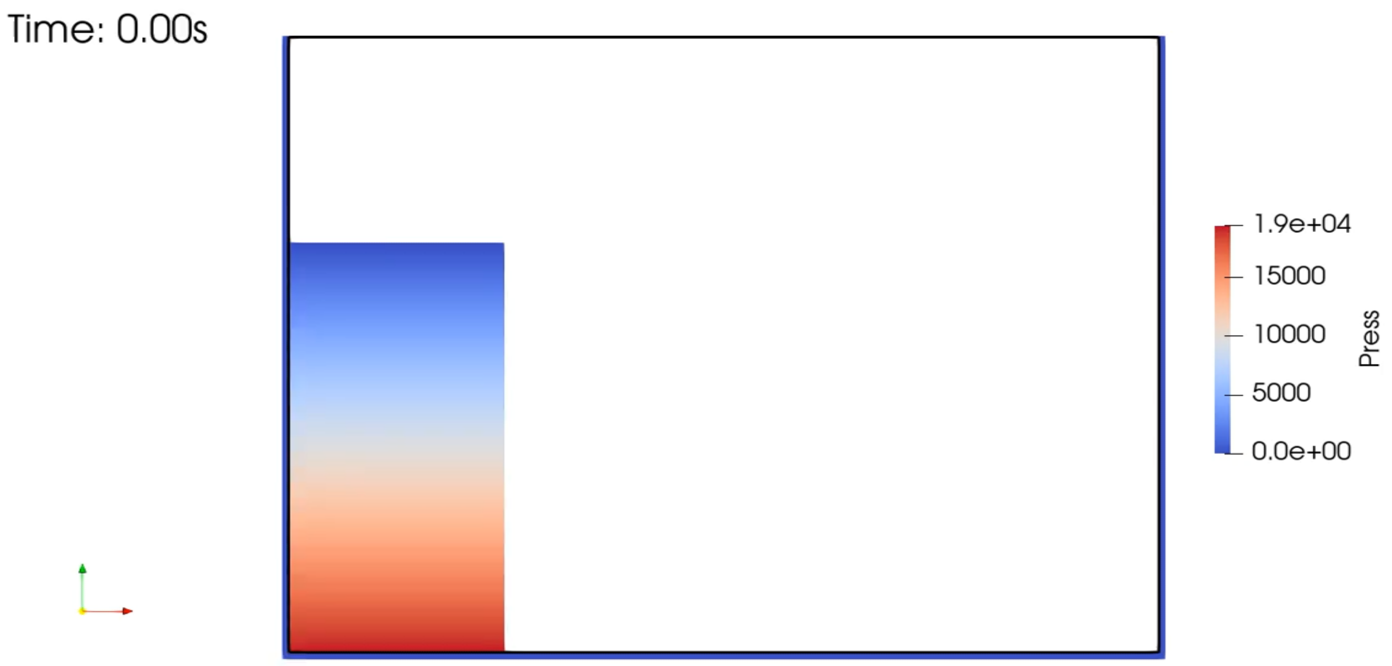
- *Molteni, D., & Colagrossi, A. (2009) *CPC*
- Ferrari, A., et al., (2009)
- Antuono, M., et al., (2012) CPC
- Green, M. D., et al., (2019). C&F
- **Fourtakas, G., et al., (2019). *C&F***



D_a is effective in reducing the high – frequency pressure oscillation (length ∝ h)

Compressibility of the WCSPH scheme (order of 1%)

- High numerical speed of sound results in very small timesteps
- We (mostly) favour a nearly incompressible flow



Courtesy of English A., 52 sec compute time on a Nvidia RTX 4090

In WC-SPH low frequency pressure wave are present

How to remove acoustic pressure

Diffusive approach (Sun et al. 2023 JCP) – similar to bulk viscosity

Added term in momentum equation

$$\frac{d\mathbf{u}_a}{dt} = -\frac{1}{\rho_a}\sum_b (P_a + P_b)\nabla W_{ab} V_b + \langle \nu_a \nabla^2 \mathbf{u}_a \rangle + \boldsymbol{F}_a^{ad}$$

$$\boldsymbol{F}_a^{ad} = \lambda \nabla(\nabla \cdot \mathbf{u}) \Rightarrow \begin{cases} \boldsymbol{F}_a^{ad} = \lambda \sum_b (\nabla \cdot \mathbf{u}_b + \nabla \cdot \mathbf{u}_a)\, \nabla W_{ab}\, V_b \\ \nabla \cdot u_k = \sum_l \mathbf{u}_{lk} \cdot \nabla W_{kl}\, V_l \end{cases}$$

Drawbacks:

- double loop summation
- free tuning term λ

How to remove acoustic pressure

Hyperbolic/parabolic approach (Fourtakas et al. 2025 C&F)

A fictitious variable ψ is introduced and coupled to $\mathbf{u}$ through momentum eq. extra term

$$\frac{d\mathbf{u}_a}{dt} = -\frac{1}{\rho_a}\sum_b (P_a + P_b)\nabla W_{ab} V_b + \langle \nu_a \nabla^2 \mathbf{u}_a \rangle - \nabla\psi$$

ψ evolves temporally according to

$$\frac{d\psi}{dt} = -c_0^2(\nabla \cdot \mathbf{u}) - \frac{\psi}{\tau}$$

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Divergence cleaning for weakly compressible smoothed particle hydrodynamics

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How to remove acoustic pressure

- Hyperbolic/parabolic approach

- Combining the equations and taking the divergence, we obtain **a wave equation** with a diffusive parabolic term:

$$\boxed{\frac{\partial^2(\nabla\cdot\mathbf{u})}{\partial t} - c_0^2\nabla^2(\nabla\cdot\mathbf{u})} + \boxed{\frac{1}{\tau}\frac{\partial(\nabla\cdot\mathbf{u})}{\partial t} = 0\,,}$$

- Tricco et al. suggested global approach for MHD:

$$\frac{1}{\tau} = \frac{\sigma c_0}{h}\,,$$

$\sigma = [0,1]$ (suggest 0.1)

- This assumes filtering will be applied based on c_0 globally
- For σ values larger than 0.3 physics might be filtered out

How to remove acoustic pressure

- Local speed of sound

$$\frac{1}{\tau} = \frac{c_a}{h} \qquad c_a = \kappa\sqrt{\frac{P_a}{\rho_a}}$$

which implies an additional restriction to the time step

$$dt_v = C\frac{h}{\max\limits_a(c_a)} \ ,$$

- Discretisation
  - SPH operators used in momentum eq:

$$\nabla\psi_a = -\sum_b \psi_{ab}\nabla W_{ab}V_b \ ,$$

  - and temporal evolution

$$\boxed{\frac{d\psi_a}{dt} = c_0^2\sum_b \mathbf{u}_{ab}\cdot\nabla W_{ab}V_b - \frac{\psi_a}{\tau} \ ,}$$

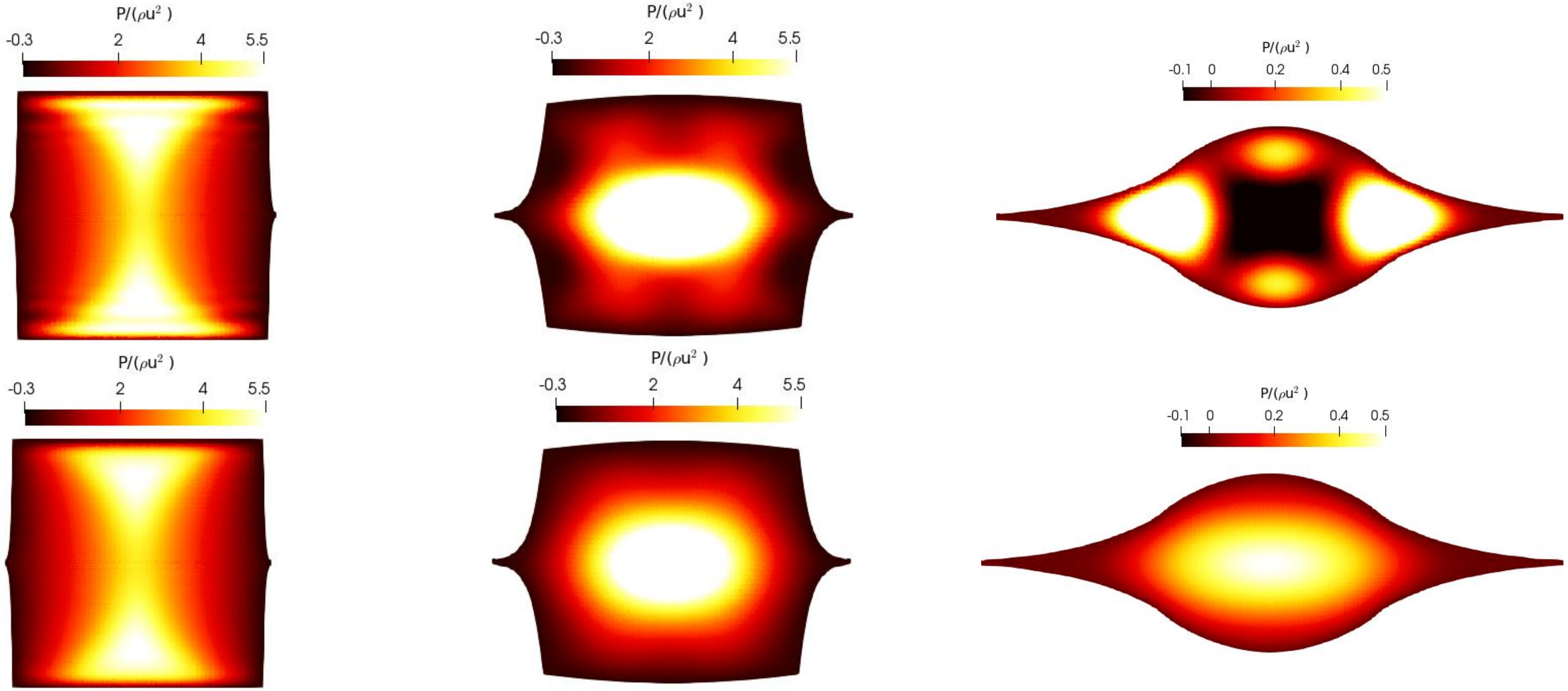
Patch test

Length $L = 2H$, with a height $H = 1$ m $\frac{L}{dx} = 200, Ma = 0.05$

Pressure field

Standard SPH

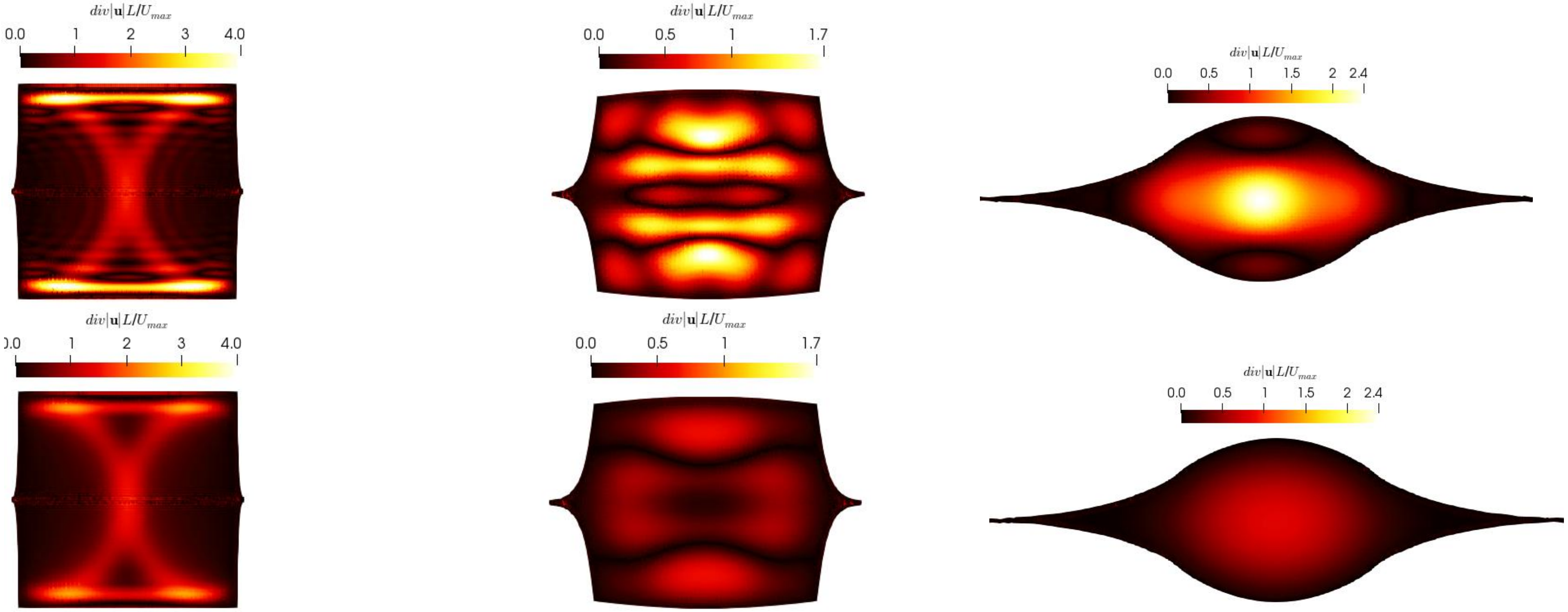
Div(**u**)-SPH



Pressure field at $t = 0.15, 0.75$ and 1 s after impact , SPH (above) for *div(**u**)*-SPH (below) for and $Ma = 0.05$

Patch test

Velocity divergence

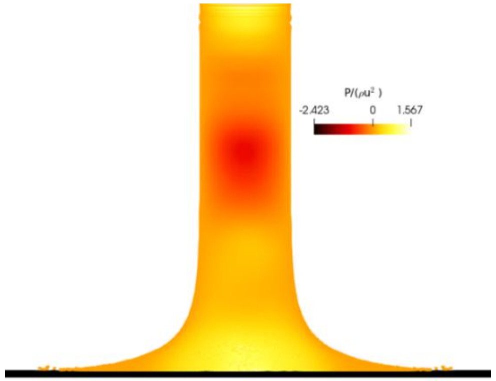


Velocity divergence at $t = 0.15, 0.75$ and 1 s after impact , SPH (above) for *div(***u***)*-SPH (below) for and $Ma = 0.05$

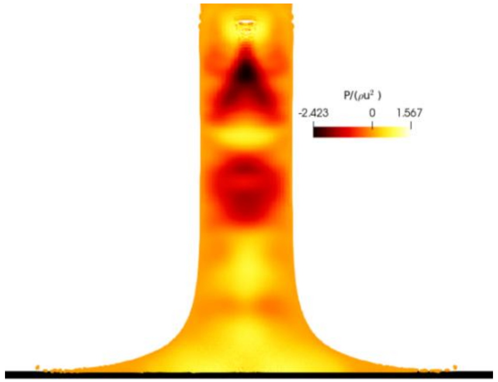
Impinging Jet

Pressure field

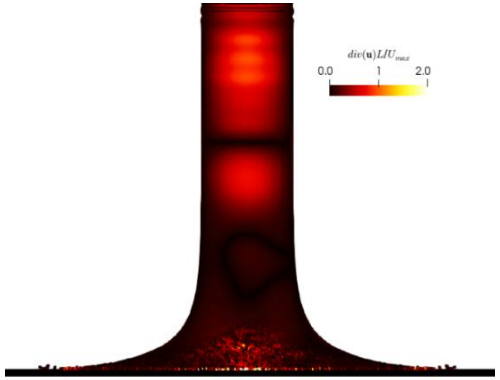
Velocity divergence



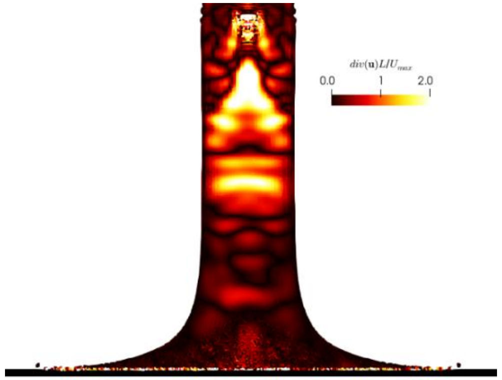
Div(**u**)-SPH



Standard SPH



Div(**u**)-SPH



Standard SPH

Pressure field at $t = 0.002$ s and $t = 0.015$ s after impact for *div(***u***)*-SPH (left) and, SPH (right) for $L/dx = 60$.

Divergence of velocity snapshots at $t = 0.002$ s and $t = 0.015$ s after impact for *div(***u***)*-SPH (left) and, SPH (right) for $D/dx = 60$

Computational cost

Test case	Time to completion overhead	Steps of simulation	
		Classical SPH	div(u)-SPH
Taylor Green vortices	1.7%	100071	100071
Patch test	18%	25285	31108
Impinging jet	-1.2%	676124	663913
Sloshing tank	-20%	346398	250779

Conclusions

- Advanced shifting improves particle distribution near the free surface.
- The ALE-SPH formulation consistently accounts for shifting through transport-velocity corrections and improves the volume conservation.
- The non-conservative pressure formulation improves accuracy for low h/dp and variable-resolution simulations.
- Divergence cleaning effectively removes acoustic pressure waves, while preserving the WCSPH framework.

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