A brief story of numerical diffusion in SPH

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Institute of Marine Engineering

Outlook of the presentation

- SPH: Origins and early applications to fluid dynamics
- Noise in WC-SPH and motivation to the use of diffusive models
- Diffusive models in WC-SPH and inspiration from other numerical schemes
- The δ-SPH
- Further diffusive schemes in SPH
- Models stemming from the δ-SPH
- Combination of diffusion and shifting techniques
- Recent advances in the use of Riemann solvers in SPH
- Diffusion and acoustic noise: the acoustic damper

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PART I

SPH: Origins and early applications to fluid dynamics

R. A. Gingold, J. J. Monaghan, Smoothed particle hydrodynamics: theory and application to non-spherical stars, *Monthly Notices of the Royal Astronomical Society*, Volume 181, Issue 3, **December 1977**

L. B. Lucy, "A numerical approach to the testing of the fission hypothesis." *Astronomical Journal,* vol. 82, **December 1977**, p. 1013-1024. 82 (1977): 1013-1024.

R. A. Gingold, J. J. Monaghan, Smoothed particle hydrodynamics: theory and application to non-spherical stars, *Monthly Notices of the Royal Astronomical Society*, Volume 181, Issue 3, **December 1977**

L. B. Lucy, "A numerical approach to the testing of the fission hypothesis." *Astronomical Journal,* vol. 82, **December 1977**, p. 1013-1024. 82 (1977): 1013-1024.

First applications in gas dynamics and astrophysics (**compressible fluids**)

1D dynamics: *shock tubes, Riemann problems, …* ⇒

oscillations of the velocity field larger than those in the density/pressure field

This issue motivated the definition of bulk viscosity to reduce the oscillations in the velocity field:

J.J Monaghan, R.A Gingold, *Shock simulation by the particle method SPH*, Journal of Computational Physics, Volume 52, Issue 2, 1983, Pages 374-389,

⇒ **definition of the artificial viscosity (later used as a real viscous term)**

First applications to hydrodynamics (**weakly-compressible fluids**)

J.J. Monaghan, *Simulating Free Surface Flows with SPH*, Journal of Computational Physics, Volume 110, Issue 2, 1994, Pages 399-406 **(received October 1992)**

2D simulations: flows past a cylinder

Takeda, Hidenori, Shoken M. Miyama, and Minoru Sekiya. "Numerical simulation of viscous flow by smoothed particle hydrodynamics." *Progress of theoretical physics* 92.5 (1994): 939-960.

J. P. Morris, P. J. Fox, and Y. Zhu. "Modeling low Reynolds number incompressible flows using SPH." *Journal of computational physics* 136.1 (1997): 214-226.
⇒ **viscosity formulation of Morris et al.(1997)**

SPH: Origins and early applications to fluid dynamics

First applications to hydrodynamics (**weakly-compressible fluids**)

J.J. Monaghan, *Simulating Free Surface Flows with SPH*, Journal of Computational Physics, Volume 110, Issue 2, 1994, Pages 399-406 **(received October 1992)**

Some issues on the pressure field are pointed out:

Morris et al. (1997)
“*The SPH dynamic pressure profile shows small local fluctuations*”

“*Once again, however, small pressure fluctuations were observed near the boundary*.”

2D simulations: flows past a cylinder

Takeda, Hidenori, Shoken M. Miyama, and Minoru Sekiya. "Nume
hydrodynamics." *Progress of theoretical physics* 92.5 (1994): 939

J. P. Morris, P. J. Fox, and Y. Zhu. "Modeling low Reynolds number incompressible flows using SPH." *Journal of computational physics* 136.1 (1997): 214-226.
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SPH: Origins and early applications to fluid dynamics

…but people did not like to talk about problems…..

Herant, M. "Dirty tricks for sph." *Memorie della Societa Astronomica Italiana* 65 (1994): 1013.

Probably, the bad name of SPH as a solver spread out in these years (i.e. 1992-2002)

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Probably, the bad name of SPH as a solver spread out in these years (i.e. 1992-2002)

In any case, different approaches were developed over the 2000's (maybe to overcome some of these issues)

Riemann solvers for particle interactions (more diffusion):

J.P. Vila, *On particle weighted methods and smooth particle hydrodynamics*, (1999), Mathematical Models and Methods in Applied Sciences 161-209

A.N. Parshikov, S.A. Medin , I.I. Loukashenko, V.A. Milekhin, *Improvements in SPH method by means of interparticle contact algorithm and analysis of perforation tests at moderate projectile velocities*, International Journal of Impact Engineering 24 (2000) 779-796

SPH for incompressible fluids (no acoustic noise)

S. Koshizuka, N. Atsushi, O. Yoshiaki, "Numerical analysis of breaking waves using the moving particle semi-implicit method." *International journal for numerical methods in fluids* 26.7 (1998): 751-769.

S.J. Cummins, M. Rudman. "An SPH projection method." *Journal of computational physics* 152.2 (1999): 584-607.

H. Gotoh, T. Sakai, Lagrangian Simulation of Breaking Waves Using Particle Method, Coastal Engineering Journal, vol 41, (1999) - Issue 3-4

Finally, problems came out

Imaeda, Y. & Inutsuka, S.I. Shear Flows in Smoothed Particle Hydrodynamics *The Astrophysical Journal*, **2002**, *569*, 501

*Evolutionary calculations of rotating gaseous flows around astrophysical objects with standard smoothed particle hydrodynamics (SPH) result in inaccurate evolutions of shear flows. The **large density errors** emerge within one dynamical time of the system…*

Colagrossi A. & Landrini M.,Numerical simulation of interfacial flows by smoothed particle hydrodynamics, Journal of Computational Physics 191 (**2003**) 448–475

*Although the gross features, e.g. the free-surface profile, of the three solutions are practically the same, it is apparent the growth of **high-frequency pressure oscillations** in solution (A), (...)*
These spurious oscillations, and their consequences, can be significantly reduced by inserting an artificial viscous term in the momentum evolution equation, as the solution (B) shows.

R.A. Dalrymple, B.D. Rogers, Numerical modeling of water waves with the SPH method, Coastal Engineering 53 (**2006**) 141-147

*(...), the application of this methodology can lead to **unphysical behaviour** at the free surface **due to** slight **density variations** being magnified by the equation of state.*

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Imaeda, Y. & Inutsuka, S.I. Shear Flows in Smoothed Particle Hydrodynamics *The Astrophysical Journal*, **2002**, *569*, 501

XSPH, artificial viscosity

Evolutionary calculations of rotating gaseous flows around astrophysical objects with standard smoothed particle hydrodynamics (SPH) result in inaccurate evolutions of shear flows. The ***large density errors*** *emerge within one dynamical time of the system…*

Colagrossi A. & Landrini M.,Numerical simulation of interfacial flows by smoothed particle hydrodynamics, Journal of Computational Physics 191 (**2003**) 448–475

density re-initialization through MLS filter

Although the gross features, e.g. the free-surface profile, of the three solutions are practically the same, it is apparent the growth of ***high-frequency pressure oscillations*** *in solution (A), (...)*
These spurious oscillations, and their consequences, can be significantly reduced by inserting an artificial viscous term in the momentum evolution equation, as the solution (B) shows.

R.A. Dalrymple, B.D. Rogers, Numerical modeling of water waves with the SPH method, Coastal Engineering 53 (**2006**) 141-147

XSPH, Shephard filtering

(...), the application of this methodology can lead to ***unphysical behaviour*** *at the free surface* ***due to*** *slight* ***density variations*** *being magnified by the equation of state.*

Noise in WC-SPH and motivation to the use of diffusive models

Governing equations:

$$\begin{cases} \dfrac{\mathrm{d}\rho}{\mathrm{d}t} = -\rho \nabla \cdot \boldsymbol{u} \\ \rho \dfrac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = -\nabla p + (\lambda + \mu)\nabla (\nabla \cdot \boldsymbol{u}) + \mu \nabla^2 \boldsymbol{u} \\ p = f(\rho) \\ \dfrac{\mathrm{d}}{\mathrm{d}t} = \dfrac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla \end{cases}$$

The fluid is assumed to be:

- **compressible**
- barotropic (density and pressure related through a state equation)

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To approximate incompressible fluids, we require the fluid to be **weakly-compressible**

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\boxed{\frac{\mathrm{d}p}{\mathrm{d}\rho} = c^2(\rho) \gg U_0^2}
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generation of acoustic waves (acoustic noise)

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this term helps against acoustic noise

The fluid is assumed to be:

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generation of acoustic waves (acoustic noise)

To approximate incompressible fluids, we require the fluid to be **weakly-compressible**

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Noise in WC-SPH and motivation to the use of diffusive models

Particle system:

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- The fluid domain is discretized in a set of moving particles
- the differential operators are replaced by their smoothed counterparts

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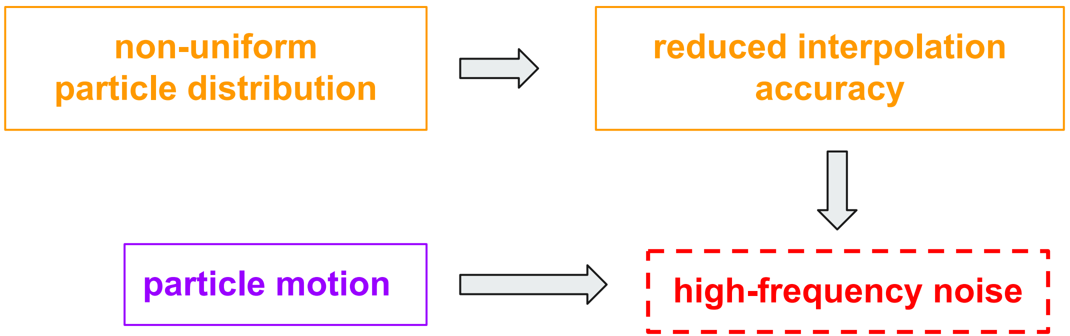

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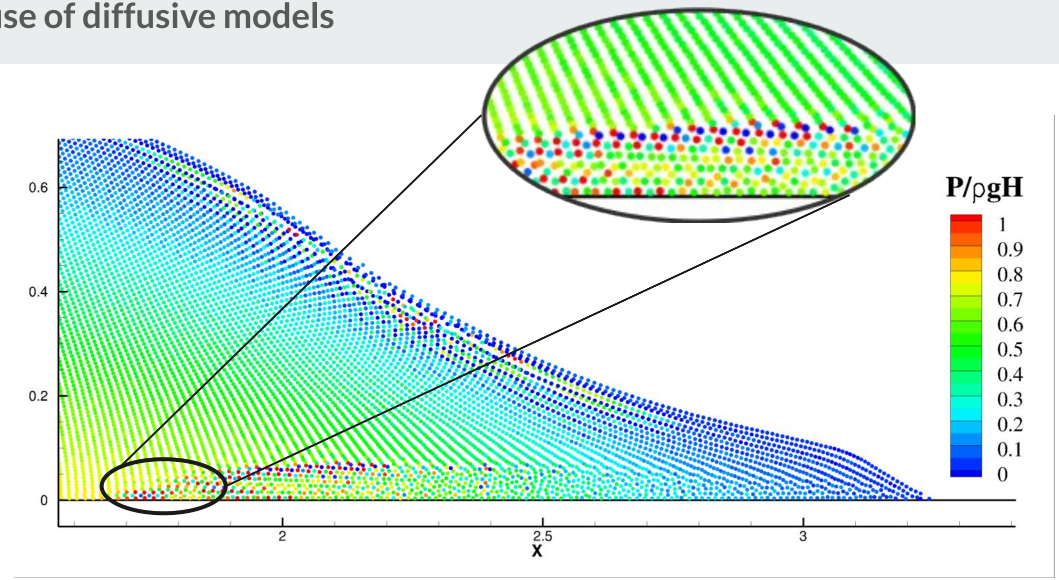
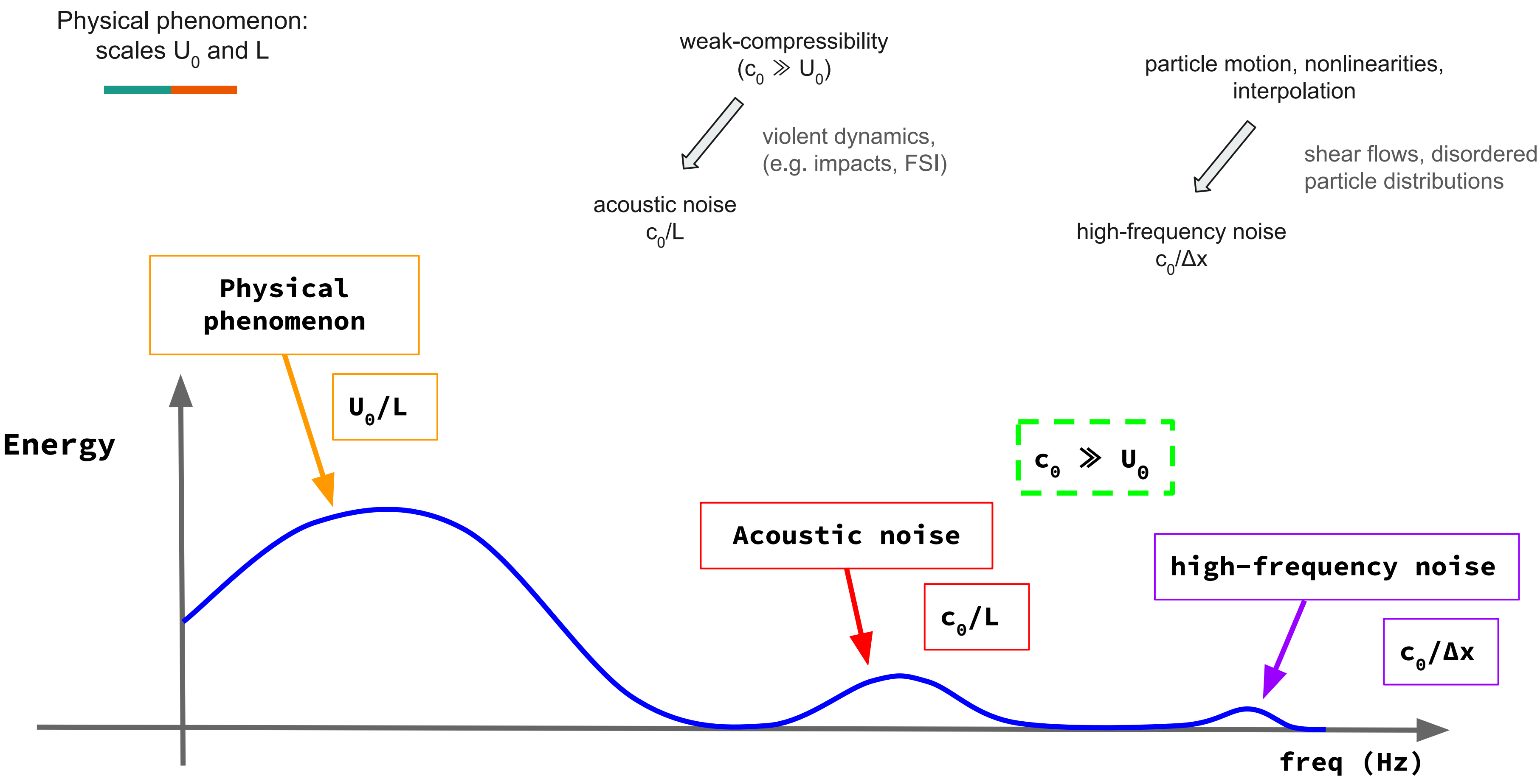


non-uniform particle distribution → reduced interpolation accuracy → high-frequency noise

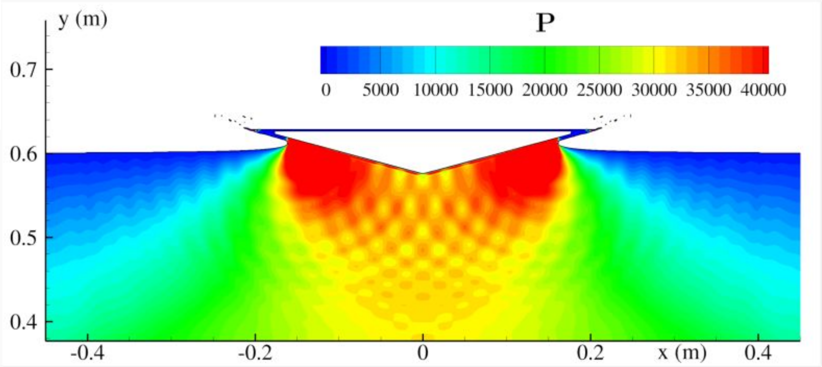
particle motion → high-frequency noise

Noise in WC-SPH and motivation to the use of diffusive models



Example of acoustic noise: the wedge entry problem

$c_0 \gg U_0$



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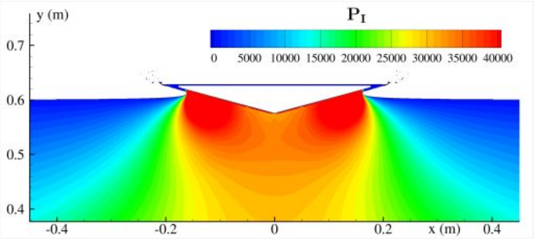
incompressible-flow pressure solution

$\nabla \cdot u = 0$



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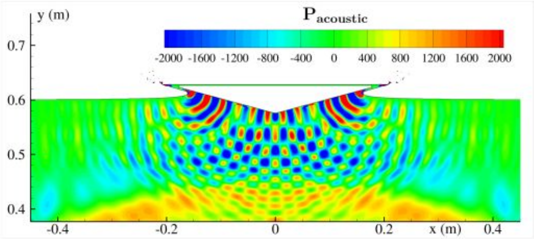


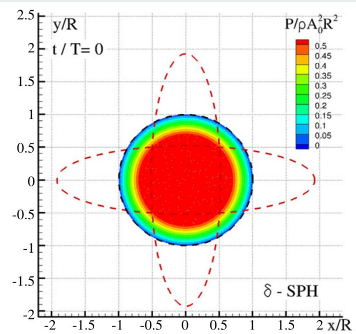
acoustic component of the pressure field

$\nabla \cdot u \neq 0$

Wedge entry problem with a deadrise angle of 15° (freely dropped from height h=0.75 m). Pressure solution at time t = 0.008s using the δ^{+}-SPH

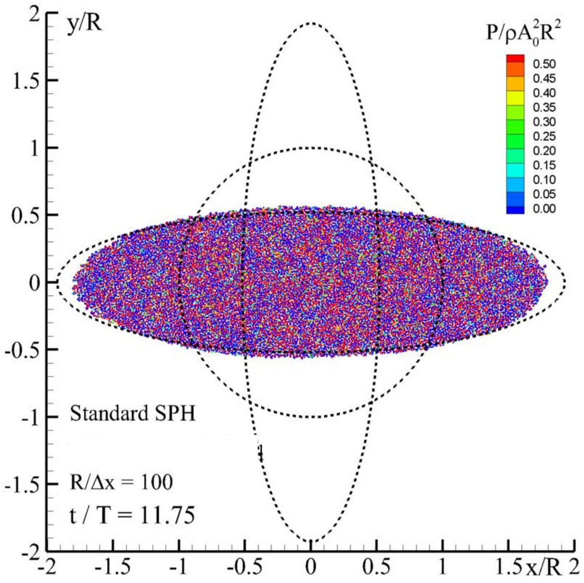
Noise in WC-SPH and motivation to the use of diffusive models

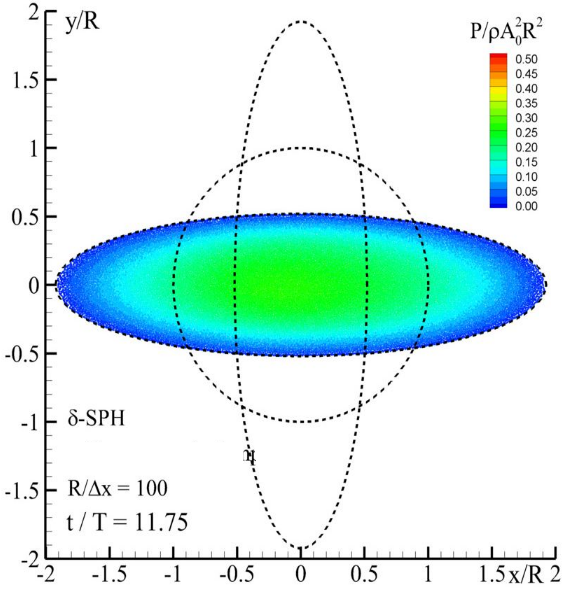
example of high-frequency noise:
inviscid oscillating drop in a central force field



$$\begin{cases} u = A(t)\,x \\ v = -A(t)\,y \end{cases}$$

central force field with potential:
$\varphi = -B^2 (x^2 + y^2)$





Noise in WC-SPH and motivation to the use of diffusive models

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- Why the pressure field is more noisy than the velocity one?
- How is possible to regularize the pressure field (at least for high-frequency noise)?

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Noise in WC-SPH and motivation to the use of diffusive models

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DIFFUSIVE TERM

$$\begin{cases} \dfrac{\mathrm{d}\rho_i}{\mathrm{d}t} = -\rho_i \langle \nabla \cdot \boldsymbol{u} \rangle_i + \mathcal{D}_i \\ \rho_i \dfrac{\mathrm{d}\boldsymbol{u}_i}{\mathrm{d}t} = -\langle \nabla p \rangle_i + \langle \nabla \cdot \mathbb{V} \rangle_i \\ p_i = f(\rho_i) \\ \dfrac{\mathrm{d}\boldsymbol{x}_i}{\mathrm{d}t} = \boldsymbol{u}_i \end{cases}$$

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- How is possible to regularize the pressure field (at least for high-frequency noise)?

The diffusive term in the continuity equation helps removing **the high-frequency noise** in the density/pressure fields (similarly to what the viscous term does in the momentum equation)

In the SPH literature the diffusion in the continuity equation has been first proposed in….

D. Molteni, A. Colagrossi, A simple procedure to improve the pressure evaluation in hydrodynamic context using the SPH, Computer Physics Communications, Volume 180, Issue 6, **2009**, Pages 861-872,
Received 14 February 2008, Accepted 2 December 2008

A. Ferrari, M. Dumbser, E. F. Toro, A. Armanini, A new 3D parallel SPH scheme for free surface flows, Computers & Fluids, Volume 38, Issue 6, **2009**, Pages 1203-1217,
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Diffusive models in WC-SPH and inspiration from other numerical schemes

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> "*Such a procedure is based on the use of a density diffusion term in the equation for the mass conservation*"

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Riemann solvers in SPH ⇒

> "*The new key idea consists of introducing a monotone upwind flux, following directly the Ben Moussa and Vila approach, but only for the density equation*"

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"Such a procedure is based on the use of a density diffusion term in the equation for the mass conservation"

Both diffusive terms approximate the Laplacian of the density field

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Riemann solvers in SPH ⟹

"The new key idea consists of introducing a monotone upwind flux, following directly the Ben Moussa and Vila approach, but only for the density equation"

Similar ideas have been proposed in....

Magneto-hydrodynamics *(similarities between the magnetic and density field equations with diffusion)*

A. Dedner, F. Kemm, D. Kroner, C.D. Munz, T. Schnitzer, M. Wesenberg,
Hyperbolic Divergence Cleaning for the MHD Equations,
Journal of Computational Physics 175, 645–673 (2002)

Thermodynamics

J. R. Clausen,
Entropically damped form of artificial compressibility for explicit simulation of incompressible flow,
PHYSICAL REVIEW E 87, 013309 (**2013**)

The temperature fluctuations are related to the density field to minimize acoustic components

The diffusive variants of Molteni & Colagrossi (2009) and Ferrari et al.(2009)

$$\begin{cases} \dfrac{\mathrm{d}\rho_i}{\mathrm{d}t} = -\rho_i \langle \nabla \cdot \boldsymbol{u} \rangle_i + \mathcal{D}_i \\ \rho_i \dfrac{\mathrm{d}\boldsymbol{u}_i}{\mathrm{d}t} = -\langle \nabla p \rangle_i + \langle \nabla \cdot \mathbb{V} \rangle_i \\ p_i = f(\rho_i) \\ \dfrac{\mathrm{d}\boldsymbol{x}_i}{\mathrm{d}t} = \boldsymbol{u}_i \end{cases}$$

Molteni & Colagrossi (2009) // Ferrari et al. (2009)

$$\mathcal{D}_i \simeq \xi\, h\, c_0\, \Delta\rho$$

h smoothing length (i.e. reference length)

c_0 reference sound speed

ξ dimensionless parameter

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Problems close to the free-surface

Molteni & Colagrossi (2009) // Ferrari et al. (2009)

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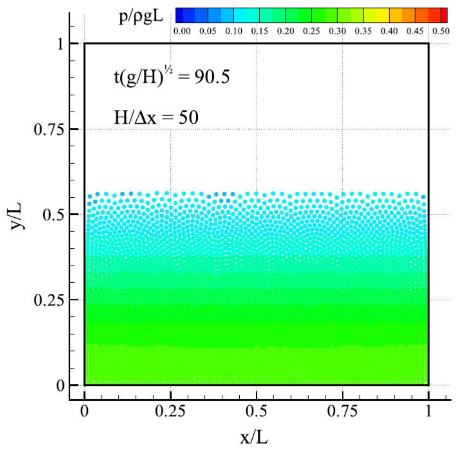


Hydrostatic solution. Left panels: SPH with the diffusive term of Ferrari et al. [11].

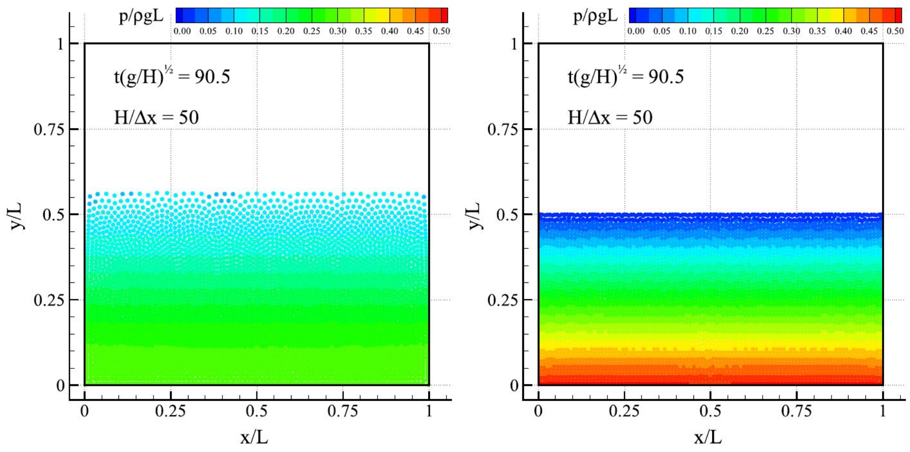
Antuono et al. (2012)

$$\begin{cases} \dfrac{d\rho_i}{dt} = -\rho_i \langle \nabla \cdot \boldsymbol{u} \rangle_i + \mathcal{D}_i \\ \rho_i \dfrac{d\boldsymbol{u}_i}{dt} = -\langle \nabla p \rangle_i + \langle \nabla \cdot \mathbb{V} \rangle_i \\ p_i = f(\rho_i) \\ \dfrac{d\boldsymbol{x}_i}{dt} = \boldsymbol{u}_i \end{cases}$$

$$\mathcal{D}_i \simeq \delta\, h^3\, c_0\, \Delta^2 \rho$$

h smoothing length (i.e. reference length)
c_0 reference sound speed
δ dimensionless parameter

Consistent close to the free-surface



Hydrostatic solution. Left panels: SPH with the diffusive term of Ferrari et al. [11]. Right panels: SPH with the diffusive term of Antuono et al. [13].

The δ-SPH

$$\mathcal{D}_i = \delta\, h\, c_0 \left[2 \sum_j \psi_{ji} \frac{(\boldsymbol{r}_j - \boldsymbol{r}_i)\cdot\nabla_i W_{ji}}{\|\boldsymbol{r}_j - \boldsymbol{r}_i\|^2} V_j \right]$$

$$\begin{cases} \dfrac{\mathrm{d}\rho_i}{\mathrm{d}t} = -\rho_i \langle \nabla \cdot \boldsymbol{u} \rangle_i + \mathcal{D}_i \\ \rho_i \dfrac{\mathrm{d}\boldsymbol{u}_i}{\mathrm{d}t} = -\langle \nabla p \rangle_i + \langle \nabla \cdot \mathbb{V} \rangle_i \\ p_i = f(\rho_i) \\ \dfrac{\mathrm{d}\boldsymbol{x}_i}{\mathrm{d}t} = \boldsymbol{u}_i \end{cases}$$

$$\psi_{ji} = \left[(\rho_j - \rho_i) - \frac{1}{2} \left(\underbrace{\langle \nabla \rho \rangle_j^L + \langle \nabla \rho \rangle_i^L}_{\text{renormalized density gradient}} \right) \cdot (\boldsymbol{r}_j - \boldsymbol{r}_i) \right]$$

renormalized density gradient

P.W. Randles, L.D. Libersky, *Smoothed particle hydrodynamics: some recent improvements and applications*, Comput. Methods Appl. Mech. Engrg. 139 (1996) 375–408

The δ-SPH preserves the mass of the fluid bulk and satisfies the conservation of both linear and angular momenta

The optimal choice is δ ≃ 0.1-0.2 [through a linear stability analysis performed in Antuono et al. (2012)]

M. D. Green, R. Vacondio, J. Peiró, *A smoothed particle hydrodynamics numerical scheme with a consistent diffusion term for the continuity equation*, Computers and Fluids 179 (2019) 632–644

"We propose to re-interpret the formulation of Antuono et al. (2012) as an approximate Riemann solver with first-order reconstruction of the density at the particle-particle interface."

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renormalized density gradient

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Time integration

$$\frac{dw}{dt} = Q + D$$

numerical diffusion

standard SPH terms

Some good ideas from Finite Volume schemes: **FROZEN DIFFUSION**

A. Jamenson, W. Schmidth, E. Turkel, *Numerical solution of the Euler equations by finite volume methods using Runge-Kutta time-stepping schemes*, in: AIAA 14th Fluid and Plasma Dynamics Conference, Palo Alto, CA, June 23–25, 1981.

Time integration

$$\frac{dw}{dt} = Q + D$$

standard SPH terms (Q)

numerical diffusion (D)

Some good ideas from Finite Volume schemes: **FROZEN DIFFUSION**

A. Jamenson, W. Schmidth, E. Turkel, *Numerical solution of the Euler equations by finite volume methods using Runge-Kutta time-stepping schemes*, in: AIAA 14th Fluid and Plasma Dynamics Conference, Palo Alto, CA, June 23–25, 1981.

fourth-order Runge-Kutta scheme with Frozen Diffusion

$$\begin{cases} w^{(0)} = w^n \\ w^{(1)} = w^{(0)} + Q(w^{(0)})\,\Delta t/2 + D(w^{(0)})\,\Delta t/2 \\ w^{(2)} = w^{(0)} + Q(w^{(1)})\,\Delta t/2 + D(w^{(0)})\,\Delta t/2 \\ w^{(3)} = w^{(0)} + Q(w^{(2)})\,\Delta t + D(w^{(0)})\,\Delta t \\ w^{(4)} = w^{(0)} + [\,Q(w^{(0)}) + 2\,Q(w^{(1)}) + 2\,Q(w^{(2)}) + Q(w^{(3)})\,]\,\Delta t/6 + D(w^{(0)})\,\Delta t \\ w^{n+1} = w^{(4)}\,. \end{cases}$$

Time integration

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Time integration

a further constraint to the time step has to be added because of the presence of diffusion in the continuity equation

$$\Delta t_c = K_c \left(\frac{h}{c_0} \right)$$

K_c = 1.3 with C2 Wendland kernel

$$\Delta t_v = \frac{1}{\alpha} \left(\frac{h}{c_0} \right)$$

$$\Delta t_a = 0.25 \min_i \sqrt{\frac{h}{\|\boldsymbol{a}_i\|}}$$

$$\boxed{\Delta t_\delta = \frac{0.44}{\delta} \left(\frac{h}{c_0} \right)}$$

$$\Delta t = \min(\Delta t_v \,, \Delta t_a \,, \Delta t_\delta \,, \Delta t_c)$$

Time integration

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$$\Delta t_\delta = \frac{0.44}{\delta}\left(\frac{h}{c_0}\right)$$

For $\delta \simeq 0.1$-0.2 this is NOT the most restrictive bound

$$\Delta t = \min(\Delta t_v\,,\, \Delta t_a\,,\, \Delta t_\delta\,,\, \Delta t_c)$$

Further diffusive schemes in SPH

R. Fatehi and M. T. Manzari, *A consistent and fast weakly compressible smoothed particle hydrodynamics with a new wall boundary condition*, Int. J. Numer. Meth. Fluids 2012; 68:905-921

Diffusion in the continuity equation as consequence of the use of different time-integration schemes in the continuity and momentum equations => the coefficient depends on the CFL

P. Ramachandran, K. Puri, *Entropically damped artificial compressibility for SPH*, Computers and Fluids 179 (2019) 579-594

Diffusion in the equation of the pressure field (in place of the density field) following the theoretical work of Clausen (2013)

G. Fourtakas, J.M. Dominguez, R. Vacondio, B.D. Rogers (2019) *Local uniform stencil (LUST) boundary condition for arbitrary 3-D boundaries in parallel smoothed particle hydrodynamics (SPH) models*. Comput Fluids 190:346-361.

Diffusion term as in Molteni & Colagrossi (2009) applied to the dynamic component of the density field (the hydrostatic component is removed)

J.J. De Courcy, T.C.S. Rendall, L. Constantin, B. Titurus, J.E. Cooper, *Incompressible δ-SPH via artificial compressibility*, Computer Methods in Applied Mechanics and Engineering 420 (2024) 116700

Diffusion in the continuity equation is obtained by using artificial compressibility
=> close analogies with the δ-SPH scheme

δ-SPH

δplus -SPH

\+ **shifting techniques**

P.N. Sun, A. Colagrossi, S. Marrone, A.M. Zhang, *The δplus-SPH model: Simple procedures for a further improvement of the SPH scheme*, Comput. Methods Appl. Mech. Engrg. 315 (2017) 25-49.

δ-ALE-SPH

\+ **Arbitrary-Lagrangian -Eulerian framework**

M. Antuono a , P.N. Sun, S. Marrone, A. Colagrossi, *The δ-ALE-SPH model: An arbitrary Lagrangian-Eulerian framework for the δ-SPH model with particle shifting technique*, Computers and Fluids 216 (2021) 104806

δLES-SPH

\+ **Large-Eddy-Simulation framework**

M Antuono, S Marrone, A Di Mascio, A Colagrossi, *Smoothed particle hydrodynamics method from a large eddy simulation perspective. Generalization to a quasi-Lagrangian model*, Physics of Fluids Volume 33 Numero 1 (2021)

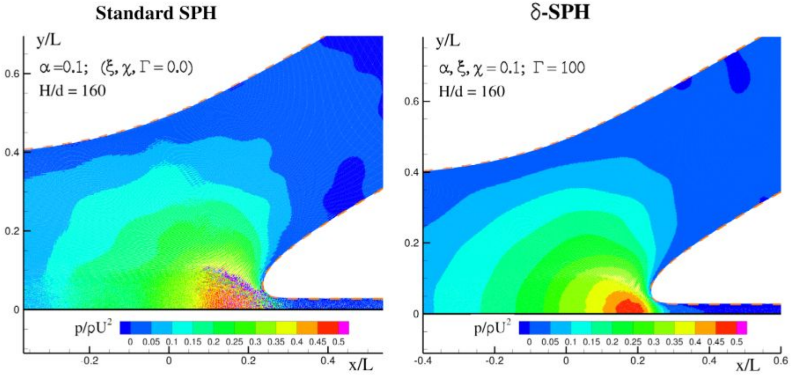
A brief story of numerical diffusion in SPH

PART II

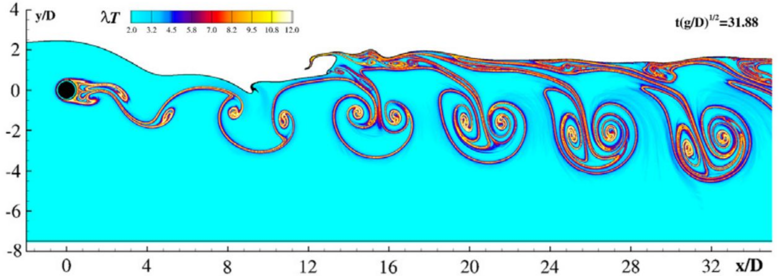
- SPH: Origins and early applications to fluid dynamics
- Noise in WC-SPH and motivation to the use of diffusive models
- Diffusive models in WC-SPH and inspiration from other numerical schemes
- The δ-SPH
- Further diffusive schemes in SPH
- Models stemming from the δ-SPH
- Combination of diffusion and shifting techniques
- Recent advances in the use of Riemann solvers in SPH
- Diffusion and acoustic noise: the acoustic damper

δ-SPH has been applied in several contexts with great success

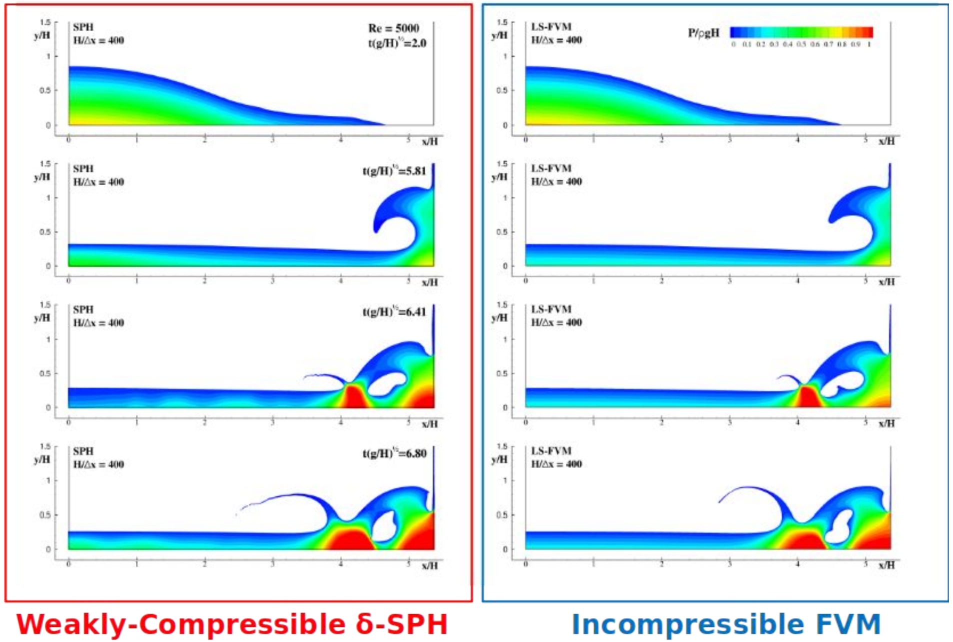
Impact of an oblique jet



Flow past cylinder below the free surface

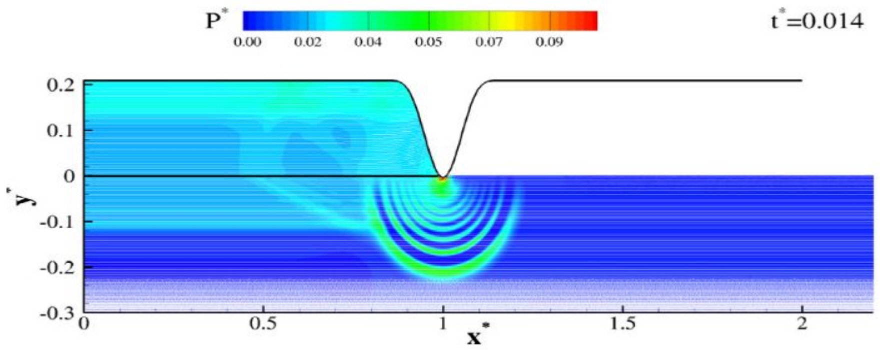


Dam-break flow

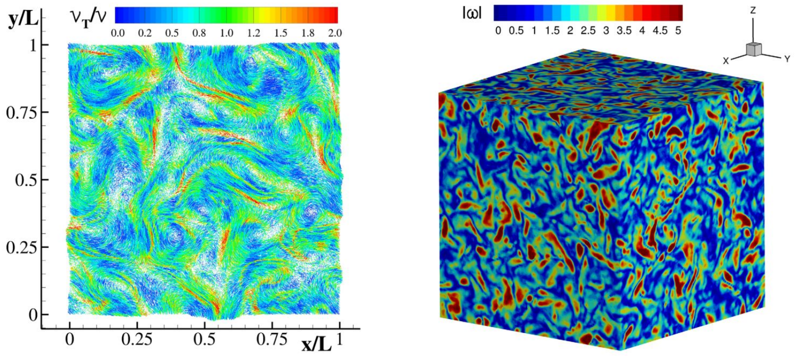


...and the δ-SPH paradigm was extended to several other contexts

Multi-phase flows



Large-Eddy Simulation models



Hammani, I., et al. "*Detailed study on the extension of the δ-SPH model to multi-phase flow*." *CMAME* 368 (2020): 113189.

Di Mascio, A., et al. "Smoothed particle hydrodynamics method from a large eddy simulation perspective." *Physics of Fluids* 29.3 (2017).

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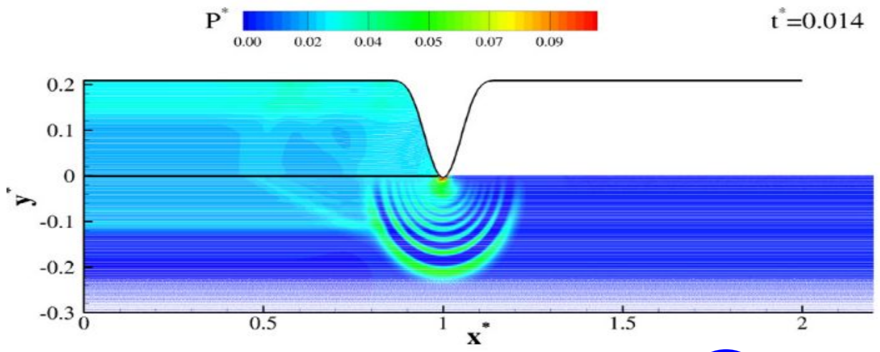

$$\frac{dV_i}{dt} = V_i \sum_i (\boldsymbol{u}_j - \boldsymbol{u}_i) \cdot \nabla W_{ij} V_j + \delta h c_{0\chi} \sum_{j \in \chi} \mathcal{D}^V_{ij} \cdot \nabla W_{ij} V_j$$

$$\mathcal{D}^V_{ij} := V_i \left[2 \left(1 - \frac{\rho_j}{\rho_i} \right) - \frac{1}{\rho_i} (\nabla^L \rho_i + \nabla^L \rho_j) \cdot \boldsymbol{r}_{ij} \right] \frac{\boldsymbol{r}_{ij}}{|\boldsymbol{r}_{ij}|^2}$$

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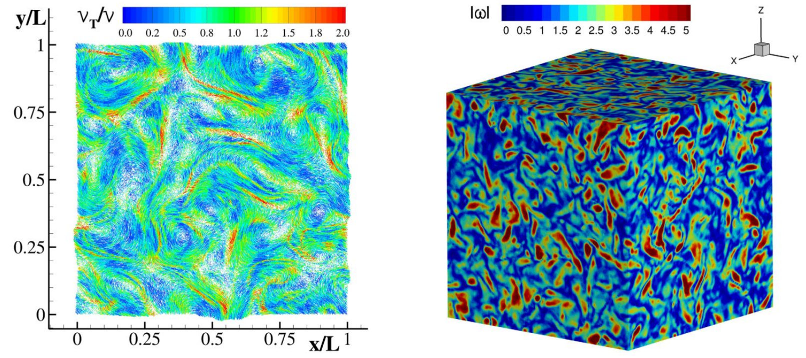
Large-Eddy Simulation models



Di Mascio, A., et al. "Smoothed particle hydrodynamics method from a large eddy simulation perspective." *Physics of Fluids* 29.3 (2017).

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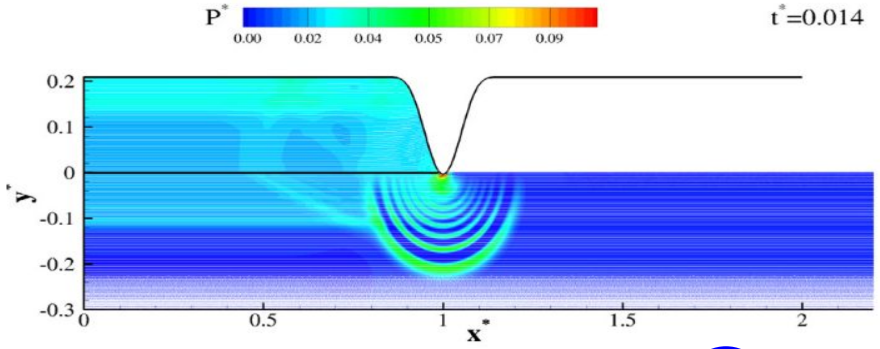

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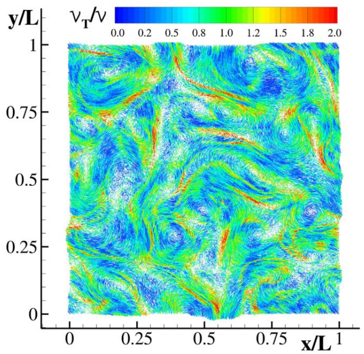
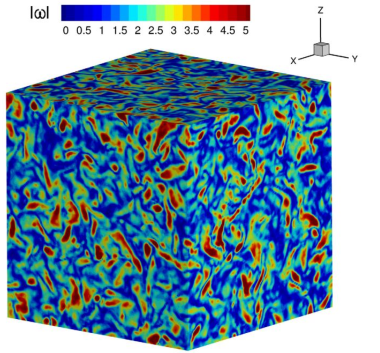
Hammani, I., et al. "*Detailed study on the extension of the δ-SPH model to multi-phase flow.*" *CMAME* 368 (2020): 113189.

Large-Eddy Simulation models



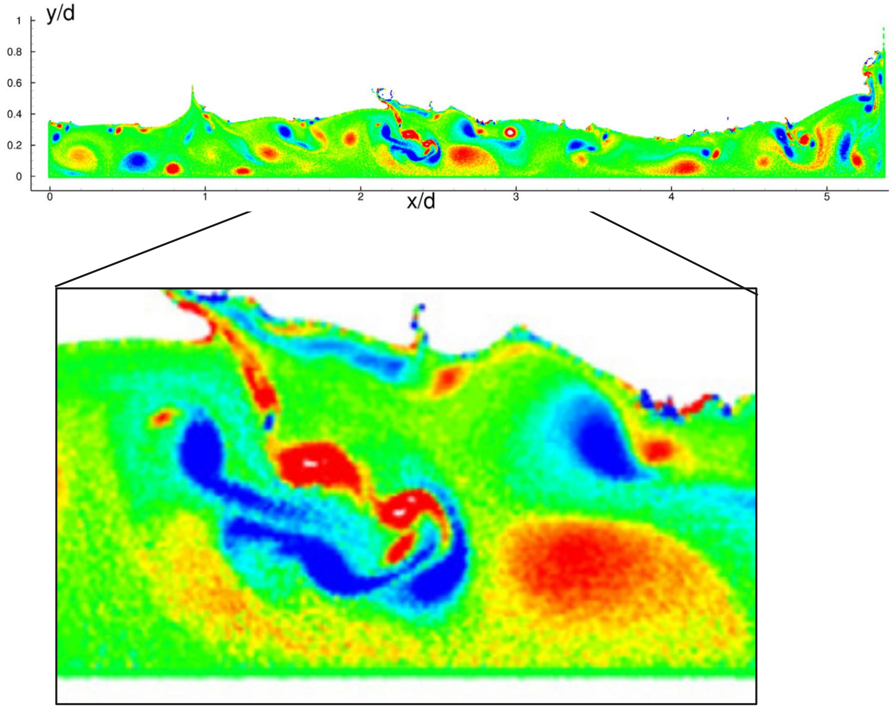


$$\frac{d\tilde{\rho}_i}{dt} = -\tilde{\rho}_i \sum_j \left(\bar{\boldsymbol{u}}_j - \bar{\boldsymbol{u}}_i \right) \cdot \nabla_i W_{ij} \, V_j + \sum_j \delta_{ij} \psi_{ji} \cdot \nabla_i W_{ij} V_j$$

depends on velocity gradients!

Di Mascio, A., et al. "Smoothed particle hydrodynamics method from a large eddy simulation perspective." *Physics of Fluids* 29.3 (2017).

Diffusive terms, however, are not a panacea…



Simulations with high vorticity and shear

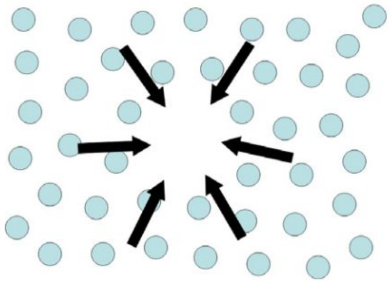
↓

Highly distorted particle distributions

↓

Higher interpolation errors and noise (remember Antuono's slide?)

Alongside the diffusive terms, a cure for particle positions: the δ⁺-SPH

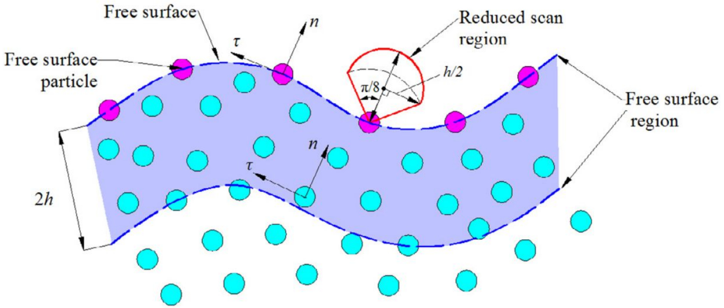


δ^{+}-SPH: correction of particle advection following Lind et al. 2012

$$\begin{cases} \boldsymbol{r}_i^* = \boldsymbol{r}_i + \delta \boldsymbol{r}_i \\ \delta \boldsymbol{r}_i := -\mathrm{CFL} \cdot \mathrm{Ma} \cdot (2\, h_{ij})^2 \cdot \sum_j \left[1 + R \left(\frac{W_{ij}}{W(\Delta x_i)} \right)^n \right] \nabla_i W_{ij}\, \varphi_{ij} \, \frac{m_j}{(\rho_i + \rho_j)} \end{cases}$$

correction for the free-surface

$$\hat{\delta \boldsymbol{r}_i} = \begin{cases} 0 & \text{if } \lambda_i < 0.4 \quad \text{and} \quad i \in \text{free-surface region} \\ (\mathbb{I} - \boldsymbol{n}_i \otimes \boldsymbol{n}_i)\, \delta \boldsymbol{r}_i & \text{if } \lambda_i \geq 0.4 \quad \text{and} \quad i \in \text{free-surface region} \\ \delta \boldsymbol{r}_i & \quad i \notin \text{free-surface region} \end{cases}$$

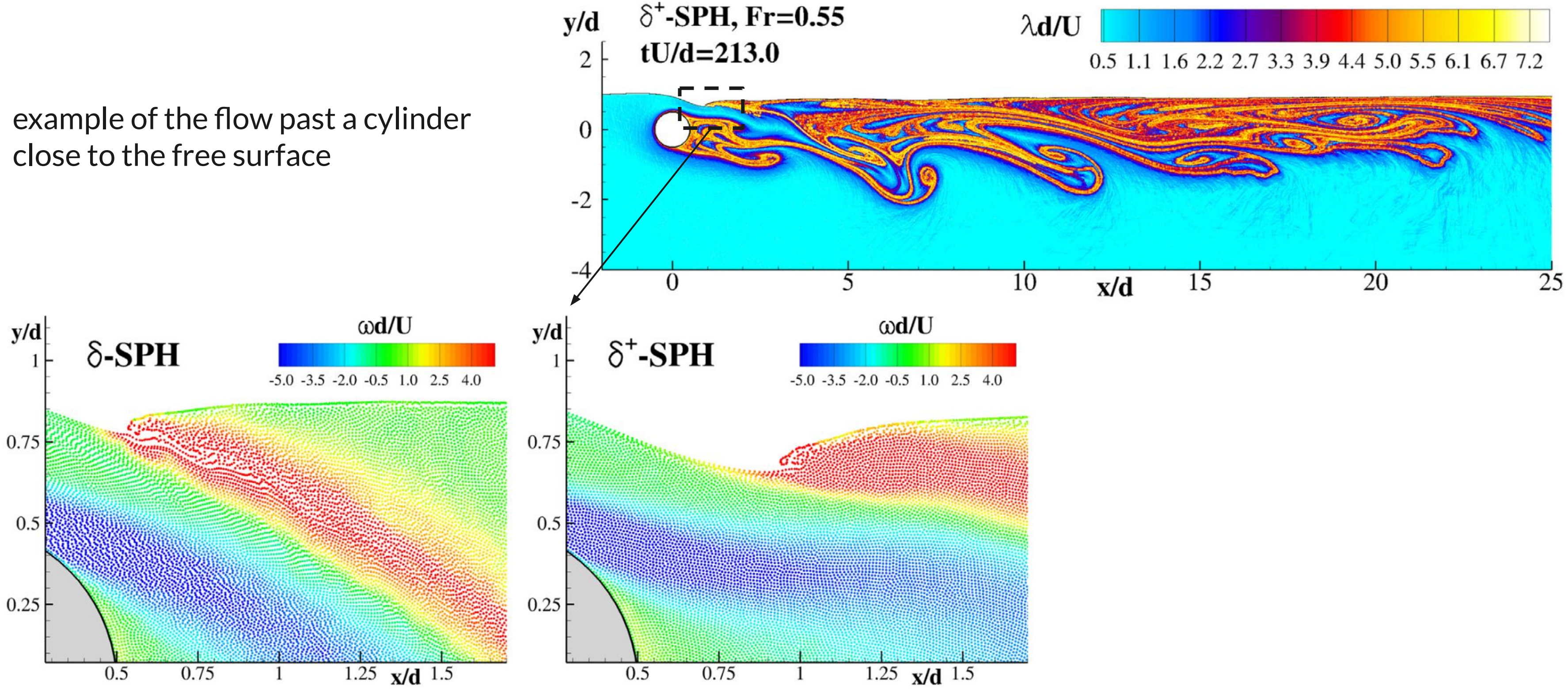


LIND, S.J. et al. “*Incompressible smoothed particle hydrodynamics for free-surface flows: A generalised diffusion-based algorithm for stability and validations for impulsive flows and propagating waves*” *JCP* (2012)

SUN, P. N., et al. “*The δ⁺-SPH model: Simple procedures for a further improvement of the SPH scheme*” *CMAME*, 2017.

In some cases it can result in a substantial improvement

example of the flow past a cylinder close to the free surface



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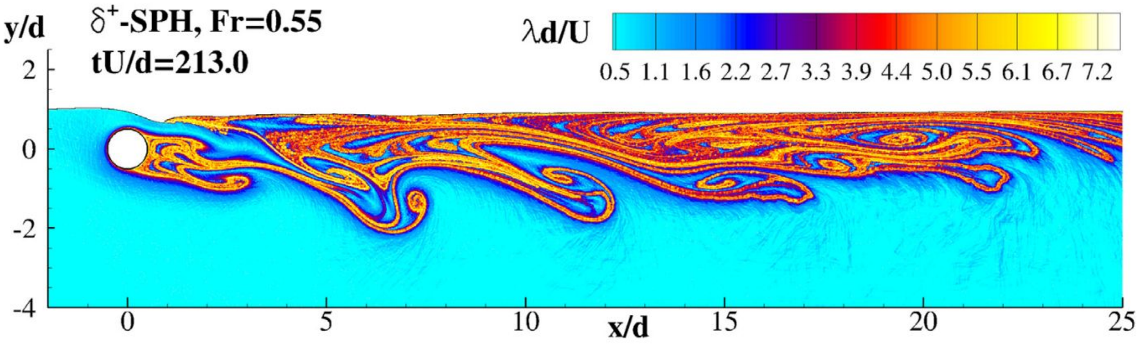

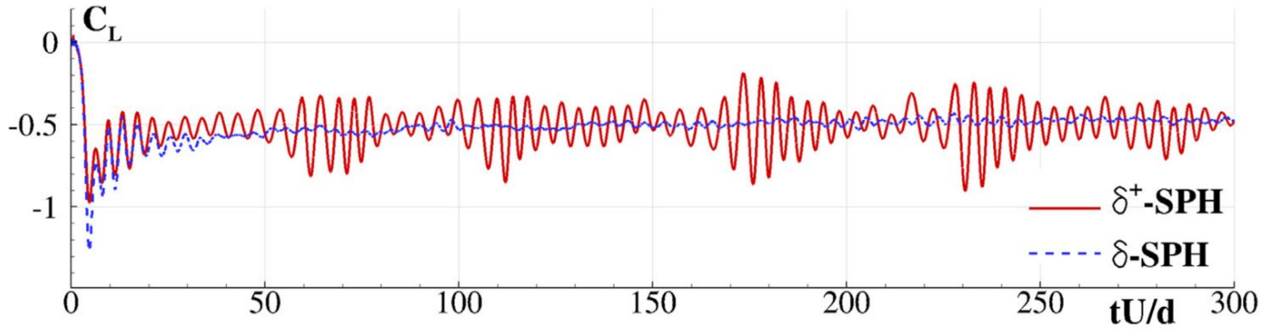

comparisons with FVM solutions confirmed δ^{+}-SPH result

COLAGROSSI, A., et al. *Viscous flow past a cylinder close to a free surface: Benchmarks with steady, periodic and metastable responses, solved by meshfree and mesh-based schemes*. *Computers & Fluids*, 2019.

Advection correction must be done in a consistent framework

Lagrangian derivative must be re-defined for a particle moving with advection velocity $(\vec{u} + \delta\vec{u})$

$$\frac{\mathrm{d}f}{\mathrm{d}t} := \frac{\partial f}{\partial t} + \nabla f \cdot (\vec{u} + \delta\vec{u})$$

Thus leading to the **δ-ALE-SPH** scheme

$$\begin{cases} \dfrac{\mathrm{d}\rho}{\mathrm{d}t} = -\rho\,\mathrm{div}(\vec{u} + \delta\vec{u}) + \mathrm{div}(\rho\,\delta\vec{u}) + \mathcal{D}^{\rho}\,, \\ \dfrac{\mathrm{d}m}{\mathrm{d}t} = m\,\dfrac{\mathrm{div}(\rho\,\delta\vec{u})}{\rho} + \mathcal{D}^{m}\,, \\ \dfrac{\mathrm{d}(m\vec{u})}{\mathrm{d}t} = m\left[-\dfrac{\nabla p}{\rho} + \dfrac{\mathrm{div}(\mathbb{T}^{\nu})}{\rho} + \vec{g} + \dfrac{\mathrm{div}(\rho\,\vec{u}\otimes\delta\vec{u})}{\rho}\right], \\ \dfrac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \vec{u} + \delta\vec{u}\,, \quad V = m/\rho\,, \quad p = c_0^2\,(\rho - \rho_0)\,. \end{cases}$$

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$$\frac{\mathrm{d}f}{\mathrm{d}t} := \frac{\partial f}{\partial t} + \nabla f \cdot (\vec{u} + \delta\vec{u})$$

$$|\delta \boldsymbol{u}| \ll |\boldsymbol{u}|$$

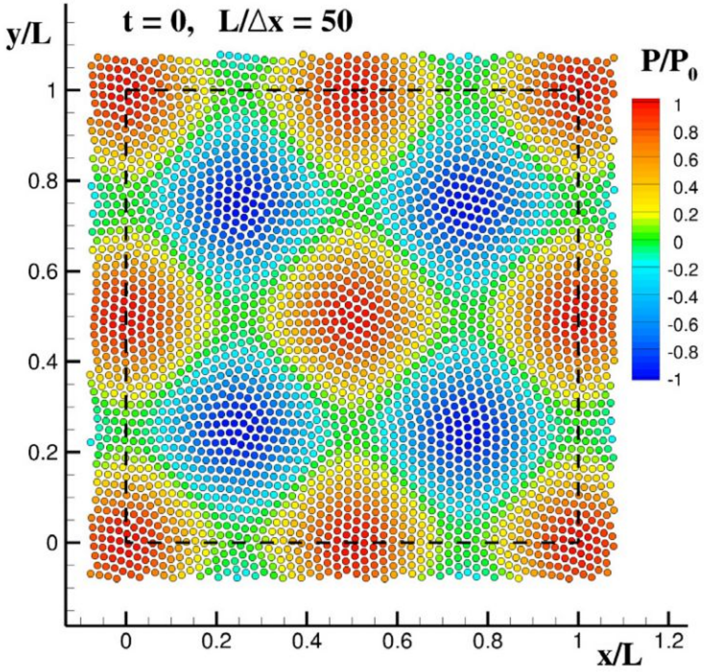
neglecting mass exchanges :
Quasi-Lagrangian scheme with constant masses

$$\begin{cases} \dfrac{\mathrm{d}\rho_i}{\mathrm{d}t} = -\rho_i \langle \mathrm{div}(\vec{u} + \delta\vec{u}) \rangle_i + \langle \mathrm{div}(\rho\delta\vec{u}) \rangle_i + \mathcal{D}_i^{\rho} \\ \dfrac{\mathrm{d}\vec{u}_i}{\mathrm{d}t} = -\dfrac{\langle \nabla p \rangle_i}{\rho_i} + \dfrac{\langle \mathrm{div}\,(\mathbb{T}^{\nu}) \rangle_i}{\rho_i} + \vec{g} + \langle \mathrm{div}(\vec{u} \otimes \delta\vec{u}) \rangle_i - \vec{u}_i \langle \mathrm{div}(\delta\vec{u}) \rangle_i \\ \dfrac{\mathrm{d}\vec{r}_i}{\mathrm{d}t} = \vec{u}_i + \delta\vec{u}_i\,, \quad V_i(t) = m_{0i} / \rho_i(t), \quad p = c_0^2\,(\rho - \rho_0)\,. \end{cases}$$

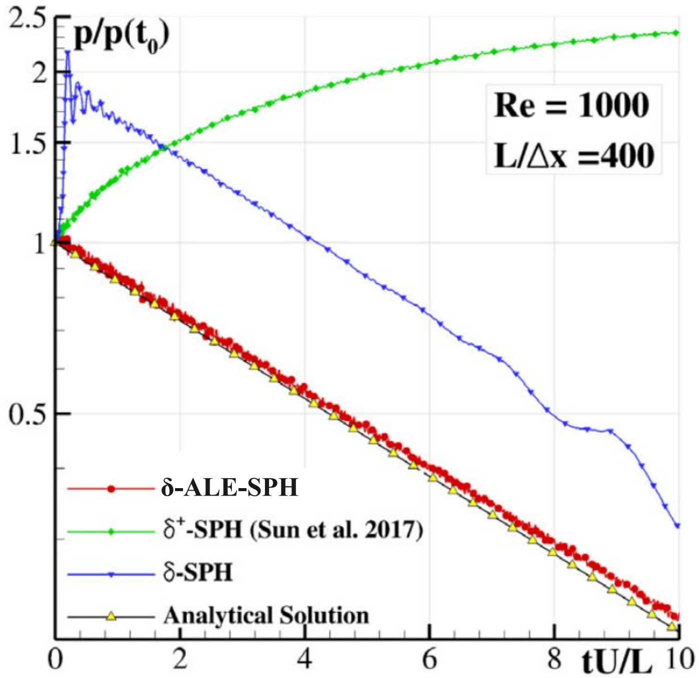
ANTUONO, M., et al. *The δ-ALE-SPH model: An arbitrary Lagrangian-Eulerian framework for the δ-SPH model with particle shifting technique*. *C&F*, 2021

Advection correction must be done in a consistent framework

Taylor-Green vortex flow

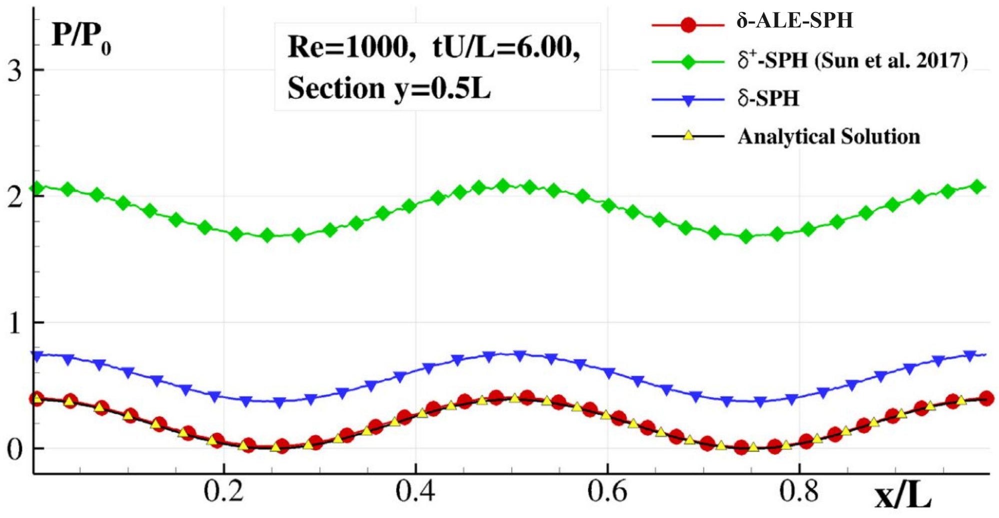


Pressure history measured in the center

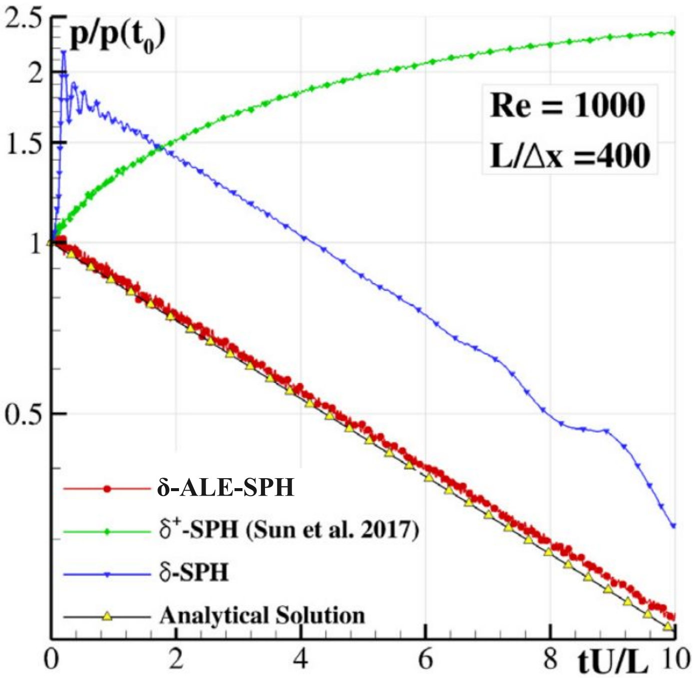


Advection correction must be done in a consistent framework

Pressure profile



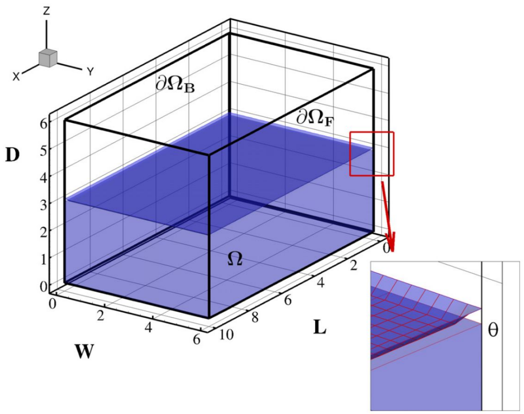
Pressure history measured in the center

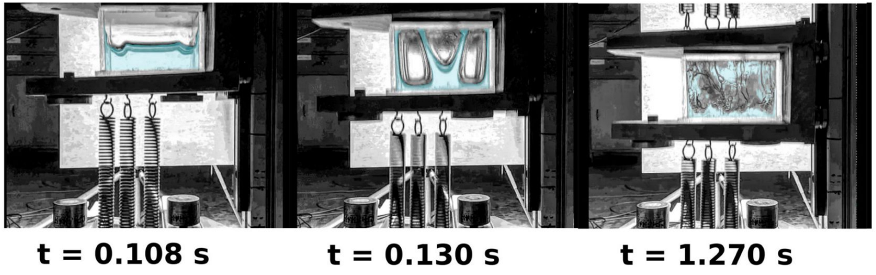


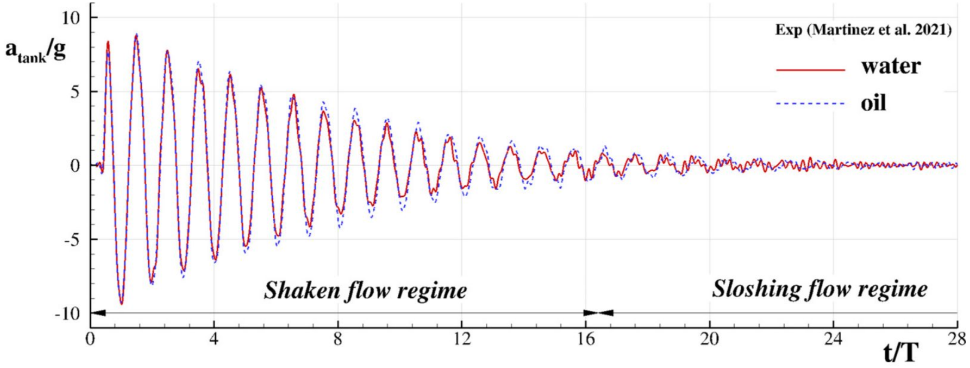
All these enhancements bring clear benefits to practical simulations

Extreme vertical sloshing in an aircraft wing tank

Flow is accelerated up to ~10g and the tank oscillates at a frequency of 6.5 Hz







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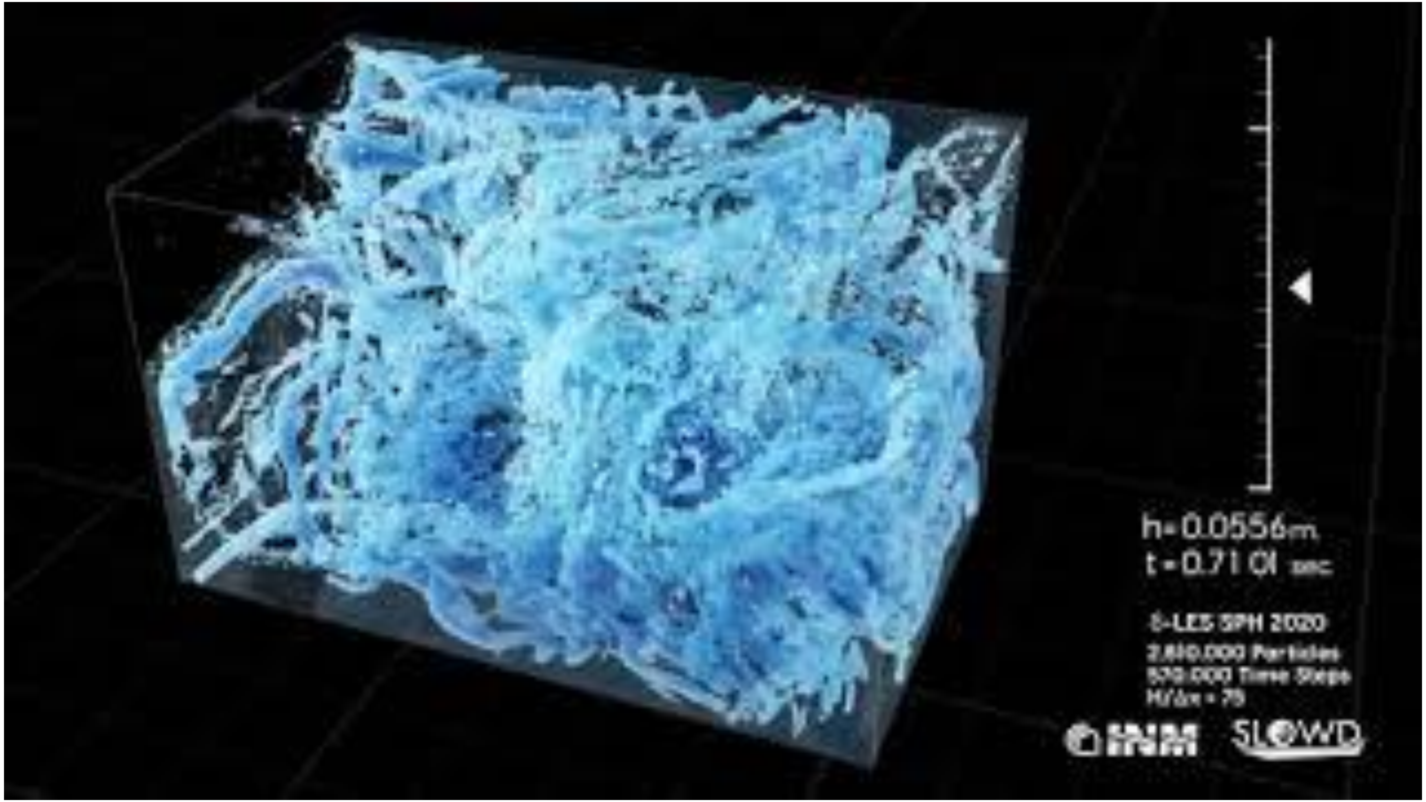
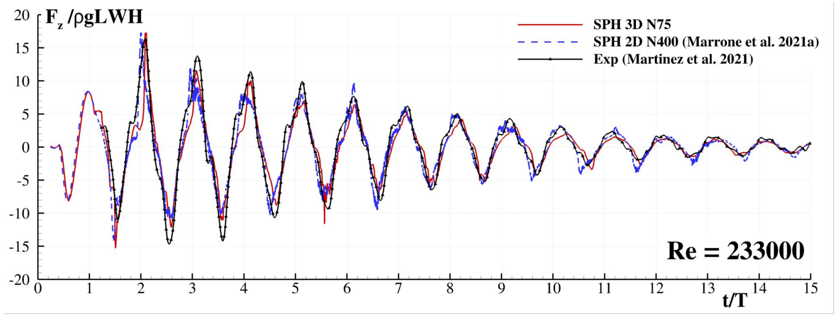


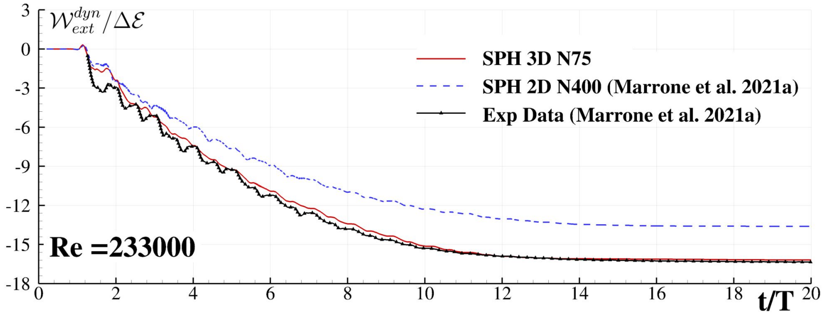
All these enhancements bring clear benefits to practical simulations

In order to accurately reproduce this phenomenon we need:

- Pressure fields free from numerical noise → Diffusive terms
- Accurate vorticity fields and low numerical diffusion → Quasi-Lagrangian scheme
- Turbulence modelling → LES model



Comparison of vertical forces



Comparison of dissipated energy

MICHEL, J., et al. "*Energy dissipation in violent three-dimensional sloshing flows induced by high-frequency vertical accelerations*". *PoF*, 2022.

A Quasi-Lagrangian scheme with higher-order diffusive terms

Primitive Variable Riemann Solver (PVRS) in place of δ-SPH diffusive terms

$$\begin{cases} \dfrac{d\rho_i}{dt} = -\rho_i \mathrm{div}(\boldsymbol{u}_i + \boldsymbol{\delta u}_i) + \mathrm{div}(\rho_i \boldsymbol{\delta u}_i) + \Theta^{\rho}_{i,Rie} \\ \rho_i \dfrac{d\boldsymbol{u}_i}{dt} = \boldsymbol{F}^p_i + \boldsymbol{F}^\mu_i + \boldsymbol{f}_i + \mathrm{div}(\rho_i \boldsymbol{u}_i \otimes \boldsymbol{\delta u}_i) + \boldsymbol{\Theta}^{\boldsymbol{u}}_{i,Rie} \\ \dfrac{d\boldsymbol{x}_i}{dt} = \boldsymbol{u}_i + \boldsymbol{\delta u}_i, \quad V_i(t) = m_i/\rho_i(t), \quad P_i = c_0^2(\rho_i - \rho_0) \end{cases}$$

$$\Theta^{\rho}_{i,Rie} = -\rho_i \sum_i (2\boldsymbol{u}_E - (\boldsymbol{u}_i + \boldsymbol{u}_j)) \cdot \nabla_i W_{ij} V_j$$

$$\boldsymbol{\Theta}^{\boldsymbol{u}}_{i,Rie} = -\sum_j [2P_E - (P_i + P_j)] \nabla_i W_{ij} V_j$$

where $\boldsymbol{u}_E$ and P_E are the solutions of the Riemann problem

A.N. Parshikov, *Application of a solution to the Riemann problem in the SPH method*, Comput. Math. Math. Phys. 1999.

A Quasi-Lagrangian scheme with higher-order diffusive terms

Primitive Variable
Riemann Solver (PVRS) in place of
δ-SPH diffusive terms

$$\begin{cases} \dfrac{d\rho_i}{dt} = -\rho_i \mathrm{div}(\boldsymbol{u}_i + \boldsymbol{\delta u}_i) + \mathrm{div}(\rho_i \boldsymbol{\delta u}_i) + \Theta^{\rho}_{i,Rie} \\ \rho_i \dfrac{d\boldsymbol{u}_i}{dt} = \boldsymbol{F}^p_i + \boldsymbol{F}^{\mu}_i + \boldsymbol{f}_i + \mathrm{div}(\rho_i \boldsymbol{u}_i \otimes \boldsymbol{\delta u}_i) + \boldsymbol{\Theta}^{\boldsymbol{u}}_{i,Rie} \\ \dfrac{d\boldsymbol{x}_i}{dt} = \boldsymbol{u}_i + \boldsymbol{\delta u}_i, \quad V_i(t) = m_i/\rho_i(t), \quad P_i = c_0^2(\rho_i - \rho_0) \end{cases}$$

It can be shown that:

$$\Theta^{\rho}_{i,Rie} = -\frac{3\,R^2}{8}\,\alpha\,\Delta\,(\nabla\cdot\boldsymbol{u}) + \frac{1}{c_0\rho_0}\,\frac{R^3}{16}\,\beta\,\Delta^2 P + \mathcal{O}\left(R^4\right)$$

$$\boldsymbol{\Theta}^{\boldsymbol{u}}_{i,Rie} = +\frac{3}{8}\,R^2\,\alpha\,\nabla\,(\Delta P) - \rho_0\,c_0\,\frac{R^3}{120}\,\gamma\,\left[\Delta^2\boldsymbol{u} + 4\,\nabla\,(\Delta\,(\nabla\cdot\boldsymbol{u}))\right] + \mathcal{O}\left(R^4\right)$$

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bi-laplacian as in δ-SPH!
(see also Green et al. 2019)

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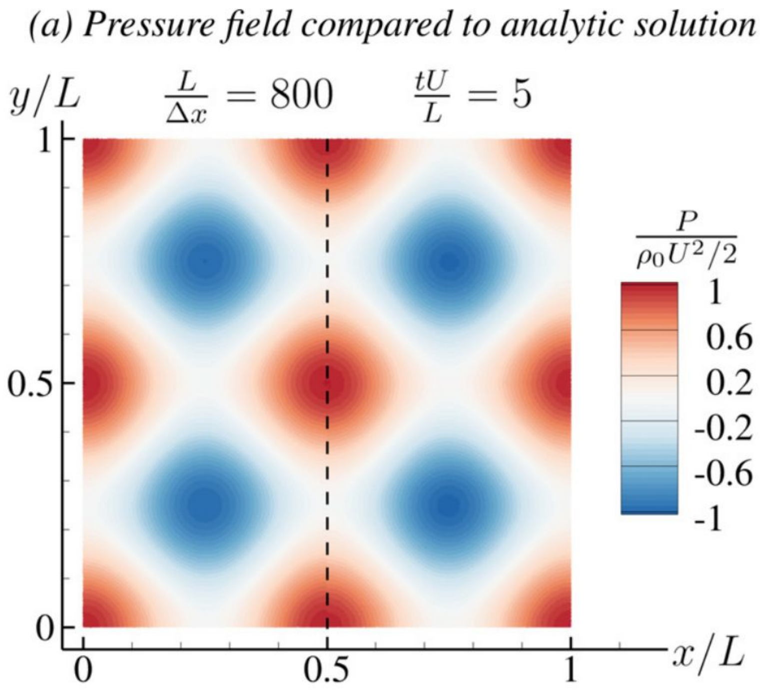
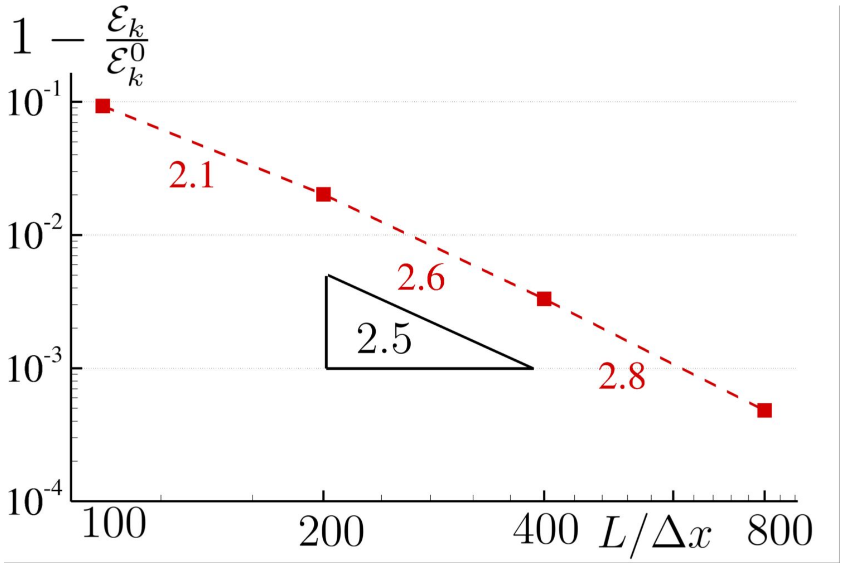
diffusive term in momentum eq. is of higher order than artificial viscosity

A Quasi-Lagrangian scheme with higher-order diffusive terms

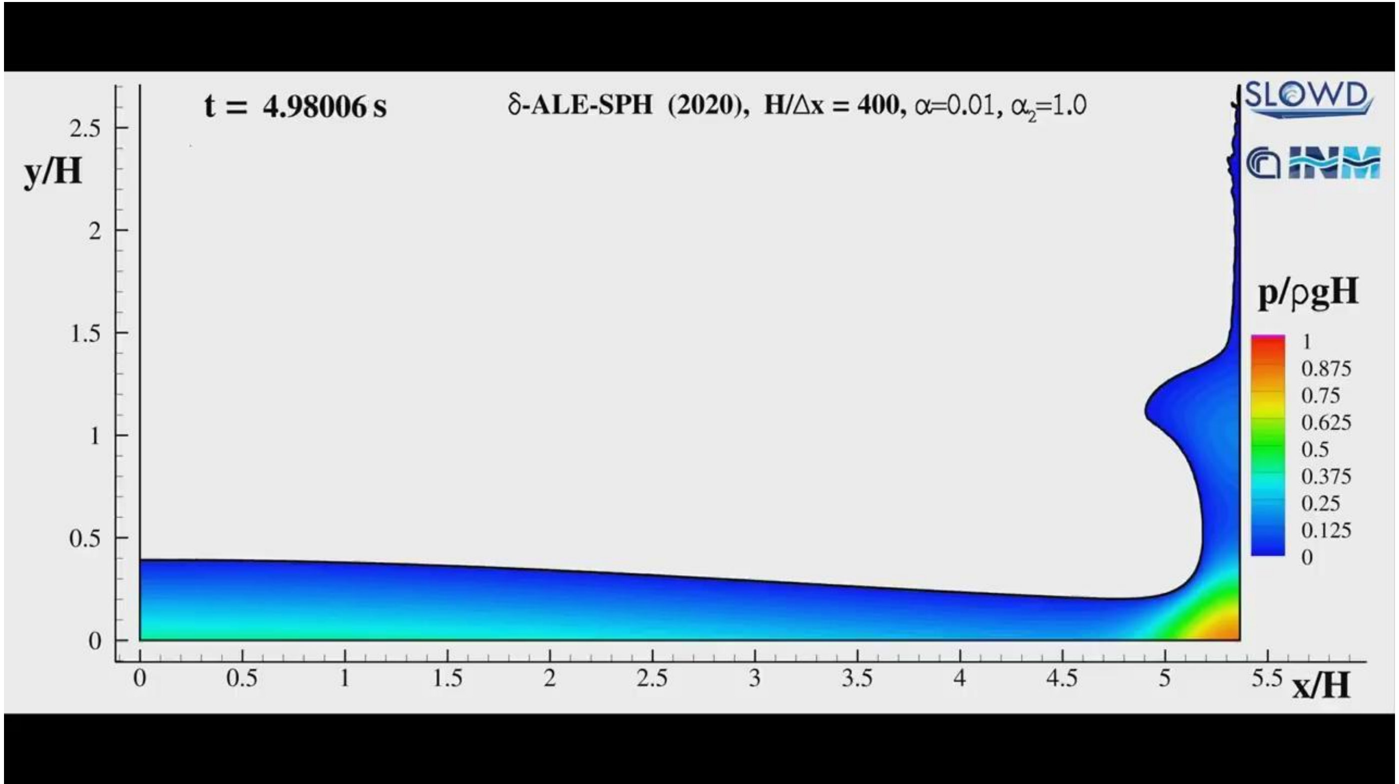


Observed convergence order between 2 and 3!

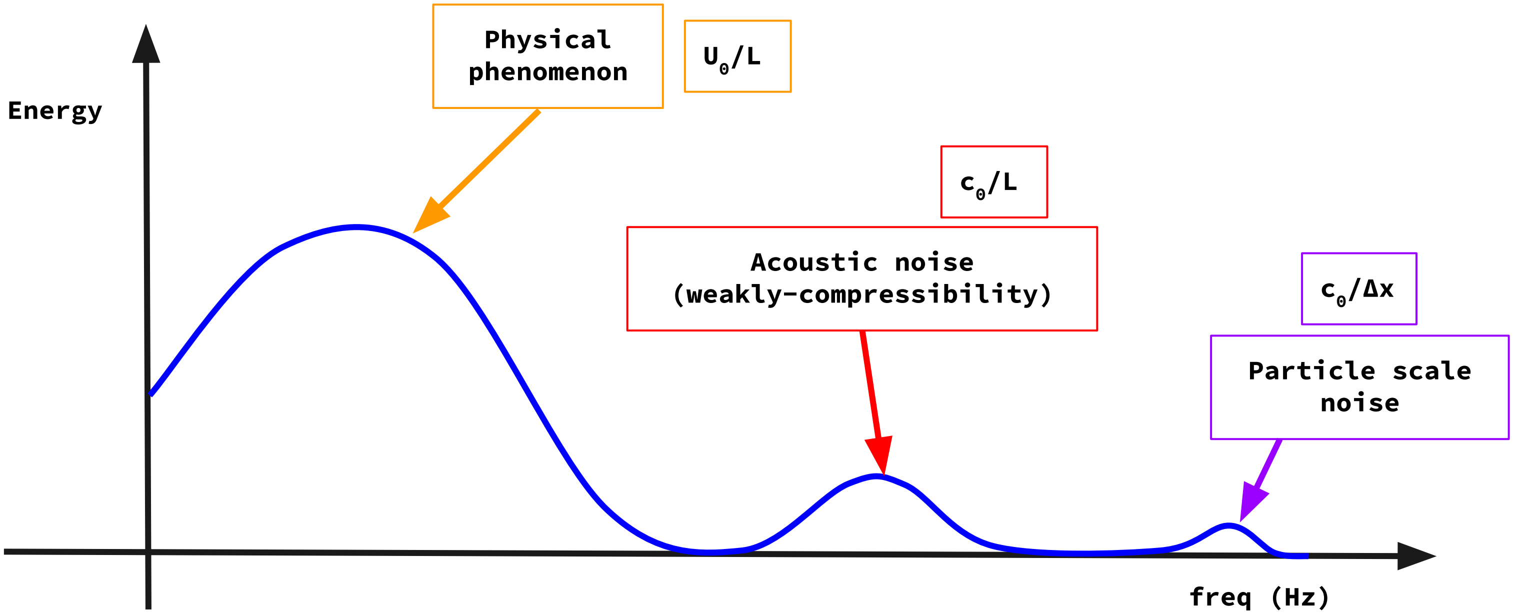


MICHEL, J., et al. "*A regularized high-order diffusive smoothed particle hydrodynamics scheme without tensile instability*" *Physics of Fluids*, 2023.

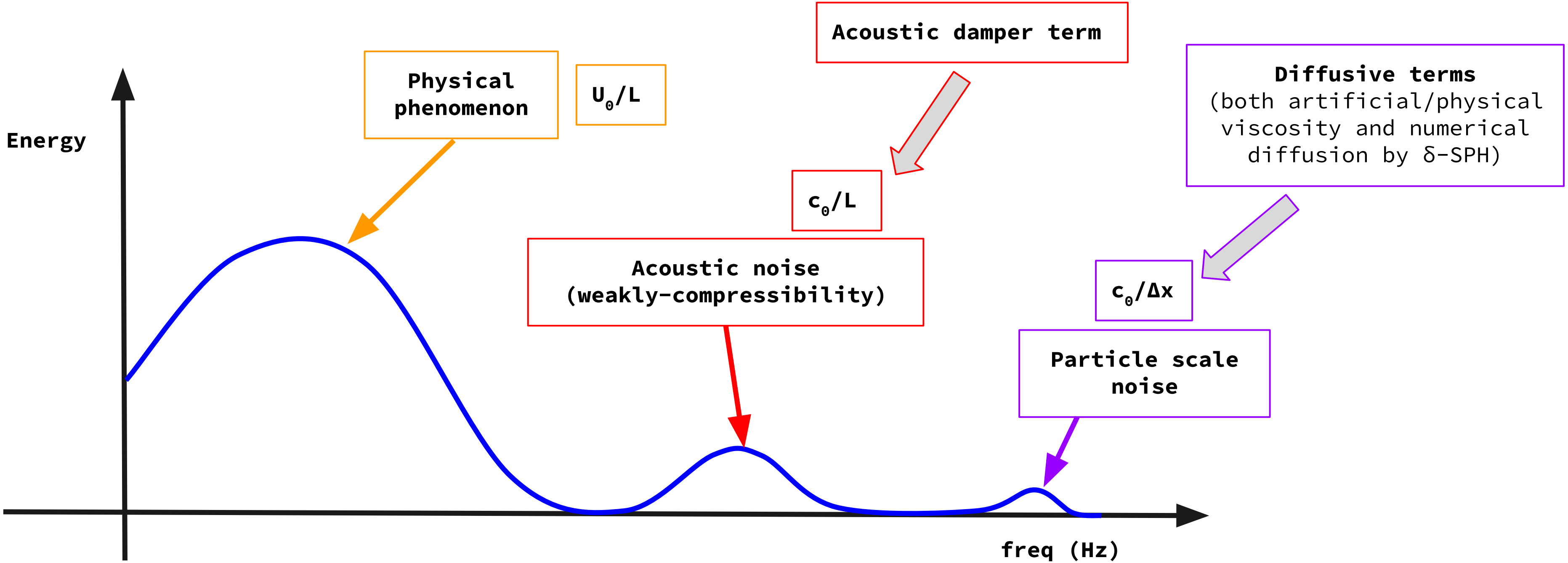
Acoustic waves sometimes are un unwanted noise on the solution



Acoustic waves can be un unwanted noise on the solution



Acoustic waves can be un unwanted noise on the solution



The δ⁺-SPH with the acoustic damper term

$$
\begin{cases}
\dfrac{\mathrm{d}\rho_i}{\mathrm{d}t} = -\rho_i \sum_j (\boldsymbol{u}_{ji} + \delta\boldsymbol{u}_{ji}) \cdot \nabla_i W_{ij} V_j \\
\qquad + \sum_j (\rho_j\, \delta\boldsymbol{u}_j + \rho_i\, \delta\boldsymbol{u}_i) \cdot \nabla_i W_{ij} V_j + \mathcal{D}_i^{\rho} \\
\rho_i \dfrac{\mathrm{d}\boldsymbol{u}_i}{\mathrm{d}t} = \boldsymbol{F}_i^{p} + \boldsymbol{F}_i^{v} + \boldsymbol{F}_i^{ad} + \rho_i \boldsymbol{g} \\
\qquad + \sum_j (\rho_j\, \boldsymbol{u}_j \otimes \delta\boldsymbol{u}_j + \rho_i\, \boldsymbol{u}_i \otimes \delta\boldsymbol{u}_i) \cdot \nabla_i W_{ij} V_j \\
\dfrac{\mathrm{d}\boldsymbol{r}_i}{\mathrm{d}t} = \boldsymbol{u}_i + \delta\boldsymbol{u}_i, \quad V_i = m_i / \rho_i, \quad p = c_0^2(\rho - \rho_0),
\end{cases}
$$

Acoustic Damper Term

The δ⁺-SPH with the acoustic damper term

$$\boldsymbol{F}_i^{ad} = \alpha_2\,\rho_0\,c_0\,h \sum_j (\dot{c}_j + \dot{c}_i)\ \nabla_i W_{ij} V_j \qquad \simeq \xi \nabla (\nabla \cdot \boldsymbol{u})$$

$$\dot{c}_k = -\dot{\rho}_k/\rho_k = \sum_l \boldsymbol{u}_{lk} \cdot \nabla_k W_{kl} V_l \qquad \simeq \nabla \cdot \boldsymbol{u}$$

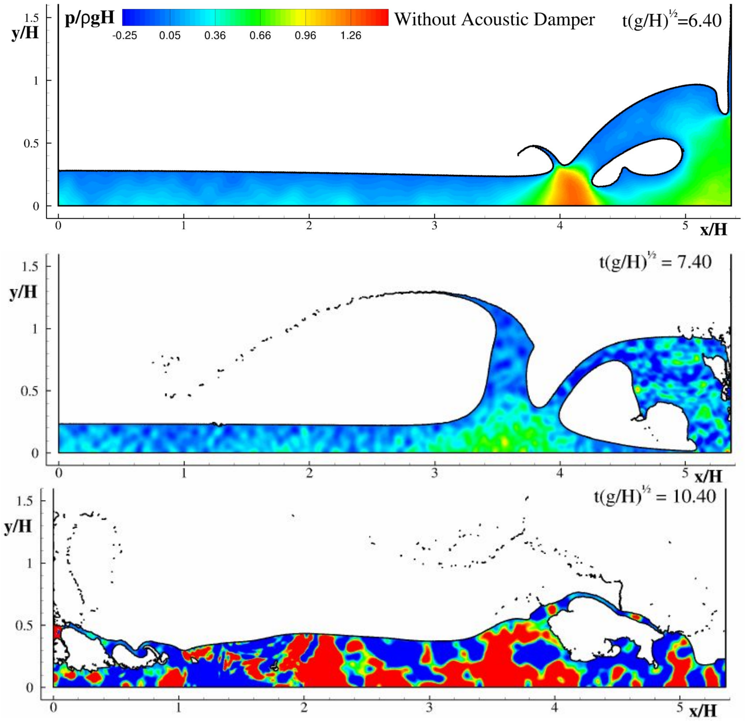
$\xi = \alpha_2\,\rho_0\,c_0\,h$ is a free bulk coefficient where α_2 is a dimensionless coefficient

Similar to the artificial viscosity of Monaghan & Gingold,
ξ decreases as h goes to zero

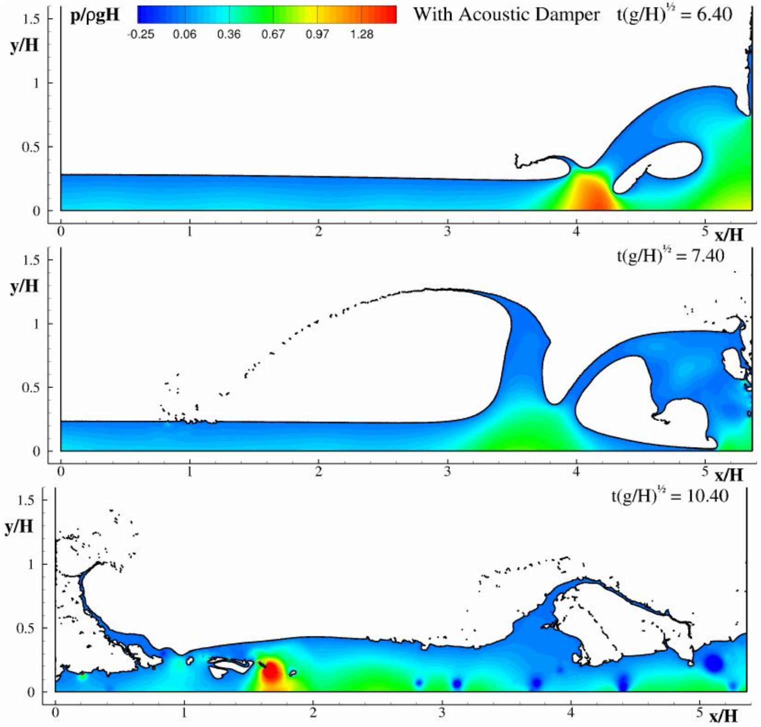
SUN, P. N., et al. *Inclusion of an acoustic damper term in weakly-compressible SPH models. JCP*, 2023.

Diffusive terms for low-frequency acoustic noise

Dam-break flow against a rectangular obstacle



without acoustic damper



with acoustic damper (α_2 = *1*)

Wrapping up

- Particle methods, such as SPH, are more prone to particle-scale noise with respect grid-based schemes because of additional degrees of freedom
- Diffusive terms are a straightforward way to alleviate this issue
- One can follow different paths to derive diffusive terms in the continuity equations, some of those are not merely numerical expedients but rely on physical bases (e.g. δ-LES)
- They are even more effective in combination with particle shifting techniques (Quasi-Lagrangian) as the latter acts on the source of the particle-scale noise
- Specific diffusive terms can be also conceived to address acoustic noise stemming from the weak-compressibility assumption
- All these ingredients allow for tackling complex and challenging problems and ongoing research continuously increases the capabilities of the SPH scheme