A brief story of numerical diffusion in SPH

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Outlook of the presentation

- SPH: Origins and early applications to fluid dynamics
- Noise in WC-SPH and motivation to the use of diffusive models
- Diffusive models in WC-SPH and inspiration from other numerical schemes
- The $\delta$-SPH
- Further diffusive schemes in SPH
- Models stemming from the $\delta$-SPH
- Combination of diffusion and shifting techniques
- Recent advances in the use of Riemann solvers in SPH
- Diffusion and acoustic noise: the acoustic damper
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First applications in gas dynamics and astrophysics (compressible fluids)

1D dynamics: shock tubes, Riemann problems, … oscillations of the velocity field larger than those in the density/pressure field

This issue motivated the definition of bulk viscosity to reduce the oscillations in the velocity field:

⇒ definition of the artificial viscosity (later used as a real viscous term)
First applications to hydrodynamics (weakly-compressible fluids)


2D simulations: flows past a cylinder


⇒ viscosity formulation of Morris et al.(1997)
First applications to hydrodynamics (weakly-compressible fluids)


2D simulations: flows past a cylinder


⇒ viscosity formulation of Morris et al.(1997)

Some issues on the pressure field are pointed out:

Morris et al. (1997)

"The SPH dynamic pressure profile shows small local fluctuations"

"Once again, however, small pressure fluctuations were observed near the boundary."
SPH: Origins and early applications to fluid dynamics

...but people did not like to talk about problems.....

Probably, the bad name of SPH as a solver spread out in these years (i.e. 1992-2002)

SPH: Origins and early applications to fluid dynamics

...but people did not like to talk about problems.....


Probably, the bad name of SPH as a solver spread out in these years (i.e. 1992-2002)

In any case, different approaches were developed over the 2000's (maybe to overcome some of these issues)

Riemann solvers for particle interactions (more diffusion):


SPH for incompressible fluids (no acoustic noise)


Finally, problems came out

Evolutionary calculations of rotating gaseous flows around astrophysical objects with standard smoothed particle hydrodynamics (SPH) result in inaccurate evolutions of shear flows. The large density errors emerge within one dynamical time of the system...

Although the gross features, e.g. the free-surface profile, of the three solutions are practically the same, it is apparent the growth of high-frequency pressure oscillations in solution (A), (...

These spurious oscillations, and their consequences, can be significantly reduced by inserting an artificial viscous term in the momentum evolution equation, as the solution (B) shows.

(....), the application of this methodology can lead to unphysical behaviour at the free surface due to slight density variations being magnified by the equation of state.


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(…), the application of this methodology can lead to unphysical behaviour at the free surface due to slight density variations being magnified by the equation of state.
Noise in WC-SPH and motivation to the use of diffusive models

The fluid is assumed to be:

- **compressible**
- barotropic (density and pressure related through a state equation)

Governing equations:

\[
\begin{align*}
\frac{d\rho}{dt} &= -\rho \nabla \cdot \mathbf{u} \\
\rho \frac{d\mathbf{u}}{dt} &= -\nabla p + (\lambda + \mu) \nabla (\nabla \cdot \mathbf{u}) + \mu \nabla^2 \mathbf{u} \\
p &= f(\rho) \\
\frac{d}{dt} &= \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla
\end{align*}
\]
The fluid is assumed to be:

- **compressible**
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To approximate incompressible fluids, we require the fluid to be **weakly-compressible**
Noise in WC-SPH and motivation to the use of diffusive models

Governing equations:

\[
\begin{align*}
\frac{d\rho}{dt} &= -\rho \nabla \cdot u \\
\rho \frac{du}{dt} &= -\nabla p + (\lambda + \mu)\nabla (\nabla \cdot u) + \mu \nabla^2 u \\
p &= f(\rho) \\
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\end{align*}
\]

The fluid is assumed to be:

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\[
\frac{dp}{d\rho} = c^2(\rho) \gg U_0^2
\]
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\end{align*}
\]

This term helps against acoustic noise
generation of acoustic waves (acoustic noise)

\[
\frac{dp}{d\rho} = c^2(\rho) \gg U_0^2
\]
Particle system:

\[
\begin{aligned}
\frac{\mathrm{d} \rho_i}{\mathrm{d} t} &= - \rho_i \langle \nabla \cdot \mathbf{u} \rangle_i \\
\rho_i \frac{\mathrm{d} \mathbf{u}_i}{\mathrm{d} t} &= - \langle \nabla p \rangle_i + \langle \nabla \cdot \nabla \rangle_i \\
p_i &= f(\rho_i) \\
\frac{\mathrm{d} x_i}{\mathrm{d} t} &= \mathbf{u}_i
\end{aligned}
\]

- The fluid domain is discretized in a set of moving particles
- the differential operators are replaced by their smoothed counterparts
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p_i &= f(\rho_i) \\
\frac{dx_i}{dt} &= u_i
\end{align*}
\]

non-uniform particle distribution \quad \Rightarrow \quad reduced interpolation accuracy

particle motion \quad \Rightarrow \quad high-frequency noise
Noise in WC-SPH and motivation to the use of diffusive models

Physical phenomenon:
scales $U_0$ and $L$

- weak-compressibility $(c_0 \gg U_0)$
- particle motion, nonlinearities, interpolation
- violent dynamics, (e.g. impacts, FSI)
- shear flows, disordered particle distributions

Physical phenomenon

Energy

- $U_0/L$
- acoustic noise $c_0/L$
- high-frequency noise $c_0/\Delta x$

freq (Hz)

Acoustic noise

$U_0/L$ $c_0/L$ $c_0/\Delta x$
Wedge entry problem with a deadrise angle of $15^\circ$ (freely dropped from height $h=0.75$ m). Pressure solution at time $t = 0.008s$ using the $\delta^+_{\text{SPH}}$ incompressible-flow pressure solution. Noise in WC-SPH and motivation to the use of diffusive models.

**Example of acoustic noise:**

the wedge entry problem

\[ \nabla \cdot \mathbf{u} = 0 \]

\[ \nabla \cdot \mathbf{u} \neq 0 \]

\[ c_\theta \gg U_\theta \]

\[ \nabla \cdot \mathbf{u} = 0 \]

\[ \nabla \cdot \mathbf{u} \neq 0 \]

\[ \text{acoustic component of the pressure field} \]

Wedge entry problem with a deadrise angle of $15^\circ$ (freely dropped from height $h=0.75$ m). Pressure solution at time $t = 0.008s$ using the $\delta^+_{\text{SPH}}$
Noise in WC-SPH and motivation to the use of diffusive models

Example of high-frequency noise:
Inviscid oscillating drop in a central force field

Central force field with potential:
\[ \varphi = -B^2 (x^2 + y^2) \]

\[
\begin{align*}
u &= A(t) x \\
v &= -A(t) y
\end{align*}
\]
Noise in WC-SPH and motivation to the use of diffusive models

Particle system:

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p_i &= f(\rho_i) \\
\frac{dx_i}{dt} &= \mathbf{u}_i
\end{aligned}
\]

- Why the pressure field is more noisy than the velocity one?
- How is possible to regularize the pressure field (at least for high-frequency noise)?
Noise in WC-SPH and motivation to the use of diffusive models

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Particle system:

\[
\begin{align*}
\frac{d\rho_i}{dt} &= -\rho_i \langle \nabla \cdot \bm{u} \rangle_i + D_i \\
\rho_i \frac{d\bm{u}_i}{dt} &= -\langle \nabla p \rangle_i + \langle \nabla \cdot \nabla \rangle_i \\
p_i &= f(\rho_i) \\
\frac{d\bm{x}_i}{dt} &= \bm{u}_i
\end{align*}
\]

- Why the pressure field is more noisy than the velocity one?
- How is possible to regularize the pressure field (at least for high-frequency noise)?

The diffusive term in the continuity equation helps removing the high-frequency noise in the density/pressure fields (similarly to what the viscous term does in the momentum equation).
In the SPH literature the diffusion in the continuity equation has been first proposed in:

D. Molteni, A. Colagrossi, A simple procedure to improve the pressure evaluation in hydrodynamic context using the SPH, Computer Physics Communications, Volume 180, Issue 6, 2009, Pages 861-872, Received 14 February 2008, Accepted 2 December 2008

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"Such a procedure is based on the use of a density diffusion term in the equation for the mass conservation"


"The new key idea consists of introducing a monotone upwind flux, following directly the Ben Moussa and Vila approach, but only for the density equation"
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“Such a procedure is based on the use of a density diffusion term in the equation for the mass conservation”


“Both diffusive terms approximate the Laplacian of the density field”

“The new key idea consists of introducing a monotone upwind flux, following directly the Ben Moussa and Vila approach, but only for the density equation”
Diffusive models in WC-SPH and inspiration from other numerical schemes

Similar ideas have been proposed in....

**Magneto-hydrodynamics** *(similarities between the magnetic and density field equations with diffusion)*

A. Dedner, F. Kemm, D. Kroner, C.D. Munz, T. Schnitzer, M. Wesenberg,
*Hyperbolic Divergence Cleaning for the MHD Equations*,

**Thermodynamics**

J. R. Clausen,
*Entropically damped form of artificial compressibility for explicit simulation of incompressible flow*,
PHYSICAL REVIEW E 87, 013309 (2013)

The temperature fluctuations are related to the density field to minimize acoustic components
The diffusive variants of Molteni & Colagrossi (2009) and Ferrari et al. (2009)

\[
\begin{align*}
\frac{d\rho_i}{dt} &= -\rho_i \langle \nabla \cdot \mathbf{u} \rangle_i + \mathcal{D}_i \\
\rho_i \frac{d\mathbf{u}_i}{dt} &= -\langle \nabla p \rangle_i + \langle \nabla \cdot \mathbf{V} \rangle_i \\
p_i &= f(\rho_i) \\
\frac{dx_i}{dt} &= \mathbf{u}_i
\end{align*}
\]

\[\mathcal{D}_i \simeq \xi h c_0 \Delta \rho\]

- \(h\) smoothing length (i.e. reference length)
- \(c_0\) reference sound speed
- \(\xi\) dimensionless parameter
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p_i &= f(\rho_i) \\
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\end{align*}
\]

Problems close to the free-surface

\[
D_i \simeq \xi h c_0 \Delta \rho
\]

- \(D_i\) smoothing length (i.e. reference length)
- \(c_0\) reference sound speed
- \(\xi\) dimensionless parameter

Hydrostatic solution. Left panels: SPH with the diffusive term of Ferrari et al. [11].
The $\delta$-SPH

\[
\frac{d\rho_i}{dt} = -\rho_i \langle \nabla \cdot \mathbf{u} \rangle_i + \mathcal{D}_i
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\]

\[
p_i = f(\rho_i)
\]

\[
\frac{dx_i}{dt} = \mathbf{u}_i
\]

Antuono et al. (2012)

\[
\mathcal{D}_i \simeq \delta h^3 c_0 \Delta^2 \rho
\]

$\delta$ dimensionless parameter

$h$ smoothing length (i.e. reference length)

$\Delta \rho$ reference sound speed

Consistent close to the free-surface

Hydrostatic solution. Left panels: SPH with the diffusive term of Ferrari et al. [11]. Right panels: SPH with the diffusive term of Antuono et al. [13].
The δ-SPH preserves the mass of the fluid bulk and satisfies the conservation of both linear and angular momenta

\[ \frac{d\rho_i}{dt} = -\rho_i \langle \nabla \cdot \mathbf{u} \rangle_i + \mathcal{D}_i \]

\[ \rho_i \frac{d\mathbf{u}_i}{dt} = -\langle \nabla p \rangle_i + \langle \nabla \cdot \mathbf{V} \rangle_i \]

\[ p_i = f(\rho_i) \]

\[ \frac{d\mathbf{x}_i}{dt} = \mathbf{u}_i \]

\[ \mathcal{D}_i = \delta h c_0 \left[ 2 \sum_j \psi_{ji} \frac{(\mathbf{r}_j - \mathbf{r}_i) \cdot \nabla_i W_{ji}}{||\mathbf{r}_j - \mathbf{r}_i||^2} V_j \right] \]

\[ \psi_{ji} = \left[ (\rho_j - \rho_i) - \frac{1}{2} \left( \langle \nabla \rho \rangle_j^L + \langle \nabla \rho \rangle_i^L \right) \cdot (\mathbf{r}_j - \mathbf{r}_i) \right] \]

renormalized density gradient


The optimal choice is \( \delta = 0.1-0.2 \) [through a linear stability analysis performed in Antuono et al. (2012)]

“We propose to re-interpret the formulation of Antuono et al. (2012) as an approximate Riemann solver with first-order reconstruction of the density at the particle-particle interface.”

The δ-SPH preserves the mass of the fluid bulk and satisfies the conservation of both linear and angular momenta

The optimal choice is \( \delta = 0.1-0.2 \) [through a linear stability analysis performed in Antuono et al. (2012)]

\[
D_i = \delta h c_0 \left[ 2 \sum_j \psi_{ji} \frac{(r_j - r_i) \cdot \nabla_i W_{ji}}{\|r_j - r_i\|^2} V_j \right]
\]

\[
\psi_{ji} = \left[ (\rho_j - \rho_i) - \frac{1}{2} \left( (\nabla \rho)^L_j + (\nabla \rho)^L_i \right) \cdot (r_j - r_i) \right]
\]

renormalized density gradient

Some good ideas from Finite Volume schemes: **FROZEN DIFFUSION**


\[
\frac{dw}{dt} = Q + D \quad \text{numerical diffusion}
\]

standard SPH terms
Some good ideas from Finite Volume schemes: FROZEN DIFFUSION


\[
\frac{dw}{dt} = Q + D \quad \text{numerical diffusion}
\]

standard SPH terms

\[
\begin{align*}
    w^{(0)} &= w^n \\
    w^{(1)} &= w^{(0)} + \frac{Q(w^{(0)})}{2} \Delta t + \frac{D(w^{(0)})}{2} \Delta t \\
    w^{(2)} &= w^{(0)} + \frac{Q(w^{(1)})}{2} \Delta t + D(w^{(0)}) \Delta t \\
    w^{(3)} &= w^{(0)} + Q(w^{(2)}) \Delta t + D(w^{(0)}) \Delta t \\
    w^{(4)} &= w^{(0)} + \left[ Q(w^{(0)}) + 2Q(w^{(1)}) + 2Q(w^{(2)}) + Q(w^{(3)}) \right] \Delta t/6 + D(w^{(0)}) \Delta t \\
    w^{n+1} &= w^{(4)}.
\end{align*}
\]
Some good ideas from Finite Volume schemes: **FROZEN DIFFUSION**


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\frac{dw}{dt} = Q + D \tag{numerical diffusion}
\]

**standard SPH terms**

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  w^{(4)} &= w^{(0)} + [Q(w^{(0)}) + 2Q(w^{(1)}) + 2Q(w^{(2)}) + Q(w^{(3)})] \Delta t/6 + D(w^{(0)}) \Delta t \\
  w^{n+1} &= w^{(4)}
\end{align*}
\]
The δ-SPH

Time integration

A further constraint to the time step has to be added because of the presence of diffusion in the continuity equation

\[ \Delta t_c = K_c \left( \frac{h}{c_0} \right) \]
\[ K_c = 1.3 \text{ with C2 Wendland kernel} \]

\[ \Delta t_v = \frac{1}{\alpha} \left( \frac{h}{c_0} \right) \]

\[ \Delta t_a = 0.25 \min_i \sqrt{\frac{h}{\|a_i\|}} \]

\[ \Delta t_\delta = \frac{0.44}{\delta} \left( \frac{h}{c_0} \right) \]

\[ \Delta t = \min(\Delta t_v, \Delta t_a, \Delta t_\delta, \Delta t_c) \]
The δ-SPH

**Time integration**

A further constraint to the time step has to be added because of the presence of diffusion in the continuity equation.

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\[ \Delta t_\delta = \frac{0.44}{\delta} \left( \frac{h}{c_0} \right) \]

For \( \delta=0.1-0.2 \) this is NOT the most restrictive bound.

\[ \Delta t = \min(\Delta t_v, \Delta t_a, \Delta t_\delta, \Delta t_c) \]
Further diffusive schemes in SPH


Diffusion in the continuity equation as consequence of the use of different time-integration schemes in the continuity and momentum equations => the coefficient depends on the CFL


Diffusion in the equation of the pressure field (in place of the density field) following the theoretical work of Clausen (2013)


Diffusion term as in Molteni & Colagrossi (2009) applied to the dynamic component of the density field (the hydrostatic component is removed)


Diffusion in the continuity equation is obtained by using artificial compressibility => close analogies with the δ-SPH scheme
## Models stemming from the δ-SPH

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<th>δplus -SPH</th>
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PART II

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δ-SPH has been applied in several contexts with great success

- Impact of an oblique jet
- Dam-break flow
- Flow past cylinder below the free surface

Weakly-Compressible δ-SPH
Incompressible FVM
The δ-SPH in free-surface flows

...and the δ-SPH paradigm was extended to several other contexts

Multi-phase flows

Large-Eddy Simulation models


The δ-SPH in free-surface flows

...and the δ-SPH paradigm was extended to several other contexts

Multi-phase flows

\[
\frac{dV_i}{dt} = V_i \sum_j (u_j - u_i) \cdot \nabla W_{ij} V_j + \delta h c_0 x \sum_{i \in y} D_{ij}^V \nabla W_{ij} V_j
\]

\[
D_{ij}^V := V_i \left[ 2 \left(1 - \frac{\rho_j}{\rho_i}\right) - \frac{1}{\rho_i} (\nabla^L \rho_i + \nabla^L \rho_j) \cdot r_{ij} \right] \frac{r_{ij}}{|r_{ij}|^2}
\]


Large-Eddy Simulation models

The δ-SPH in free-surface flows

...and the δ-SPH paradigm was extended to several other contexts

Multi-phase flows

\[ \frac{dV_i}{dt} = V_i \sum_j (u_j - u_i) \cdot \nabla W_{ij} V_j + \delta h c_{0x} \sum_{i \in Y} D_{ij}^V \nabla W_{ij} V_j \]

\[ D_{ij}^V := V_i \left[ 2 \left( 1 - \frac{\rho_j}{\rho_i} \right) - \frac{1}{\rho_i} (\nabla^L \rho_i + \nabla^L \rho_j) \cdot r_{ij} \right] \frac{r_{ij}}{|r_{ij}|^2} \]


Large-Eddy Simulation models

\[ \frac{d\tilde{\rho}_i}{dt} = -\tilde{\rho}_i \sum_j (\tilde{u}_j - \tilde{u}_i) \cdot \nabla_i W_{ij} V_j + \sum_j \delta_{ij} \psi_j \cdot \nabla_i W_{ij} V_j \]

Formulation in a quasi-Lagrangian framework

Diffusive terms, however, are not a panacea...

Simulations with high vorticity and shear

Highly distorted particle distributions

Higher interpolation errors and noise (remember Antuono’s slide?)
Alongside the diffusive terms, a cure for particle positions: the $\delta^+$-SPH

$\delta^+$-SPH: correction of particle advection following Lind et al. 2012

$$\begin{cases}
    r_i^* = r_i + \delta r_i \\
    \delta r_i := -\text{CFL} \cdot \text{Ma} \cdot (2 h_{ij})^2 \cdot \sum_j \left[ 1 + R \left( \frac{W_{ij}}{W(\Delta x_i)} \right)^n \right] \nabla_i W_{ij} \varphi_{ij} \frac{m_j}{(\rho_i + \rho_j)}
\end{cases}$$

correction for the free-surface

$$\delta^* r_i = \begin{cases}
    0 & \text{if } \lambda_i < 0.4 \text{ and } i \in \text{free-surface region} \\
    (\mathbb{I} - n_i \otimes n_i) \delta r_i & \text{if } \lambda_i \geq 0.4 \text{ and } i \in \text{free-surface region} \\
    \delta r_i & i \notin \text{free-surface region}
\end{cases}$$


Formulation in a quasi-lagrangian framework

In some cases it can result in a substantial improvement

element of the flow past a cylinder close to the free surface
Formulation in a quasi-lagrangian framework

In some cases it can result in a substantial improvement

example of the flow past a cylinder close to the free surface

comparisons with FVM solutions confirmed $\delta^+$-SPH result

Formulation in a quasi-lagrangian framework

Advection correction must be done in a consistent framework

Lagrangian derivative must be re-defined for a particle moving with advection velocity \((\vec{u} + \delta \vec{u})\)

\[
\frac{df}{dt} := \frac{\partial f}{\partial t} + \nabla f \cdot (\vec{u} + \delta \vec{u})
\]

Thus leading to the δ-ALE-SPH scheme

\[
\begin{align*}
\frac{d\rho}{dt} &= -\rho \, \text{div}(\vec{u} + \delta \vec{u}) + \text{div}(\rho \, \delta \vec{u}) + \mathcal{D}^\rho, \\
\frac{dm}{dt} &= m \, \frac{\text{div}(\rho \, \delta \vec{u})}{\rho} + \mathcal{D}^m, \\
\frac{d(m\vec{u})}{dt} &= m \left[ -\frac{\nabla p}{\rho} + \frac{\text{div}(\mathbf{T}^v)}{\rho} + \vec{g} + \frac{\text{div}(\rho \, \vec{u} \otimes \delta \vec{u})}{\rho} \right], \\
\frac{d\vec{r}}{dt} &= \vec{u} + \delta \vec{u}, \quad V = m / \rho, \quad p = c_0^2 (\rho - \rho_0).
\end{align*}
\]
Formulation in a quasi-lagrangian framework

Advection correction must be done in a consistent framework

Lagrangian derivative must be re-defined for a particle moving with advection velocity \((\vec{u} + \delta\vec{u})\)

\[
\frac{df}{dt} := \frac{\partial f}{\partial t} + \nabla f \cdot (\vec{u} + \delta\vec{u})
\]

Thus leading to the \(\delta\text{-ALE-SPH}\) scheme

\[
\begin{align*}
\frac{d\rho}{dt} &= -\rho \text{ div}(\vec{u} + \delta\vec{u}) + \text{div}(\rho \delta\vec{u}) + \mathcal{D}^\rho, \\
\frac{dm}{dt} &= m \frac{\text{div}(\rho \delta\vec{u})}{\rho} + \mathcal{D}^m, \\
\frac{d(m\vec{u})}{dt} &= m \left[ -\frac{\nabla p}{\rho} + \frac{\text{div}(\mathcal{T}^v)}{\rho} + \vec{g} + \frac{\text{div}(\rho \vec{u} \otimes \delta\vec{u})}{\rho} \right], \\
\frac{dr}{dt} &= \vec{u} + \delta\vec{u}, \quad V = m / \rho, \quad p = c_0^2 (\rho - \rho_0).
\end{align*}
\]
Formulation in a quasi-lagrangian framework

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\frac{d\vec{r}}{dt} &= \vec{u} + \delta \vec{u}, \quad V = \frac{m}{\rho}, \quad p = c_0^2 (\rho - \rho_0).
\end{align*}
\]
Formulation in a quasi-lagrangian framework

Advection correction must be done in a consistent framework

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\[
\frac{df}{dt} := \frac{\partial f}{\partial t} + \nabla f \cdot (\vec{u} + \delta \vec{u})
\]

\(|\delta \vec{u}| \ll |\vec{u}|\)

Neglecting mass exchanges:

Quasi-Lagrangian scheme with constant masses

\[
\begin{align*}
\frac{d\rho_i}{dt} &= -\rho_i \langle \text{div}(\vec{u} + \delta \vec{u}) \rangle_i + \langle \text{div}(\rho \delta \vec{u}) \rangle_i + D_i^\rho \\
\frac{d\vec{u}_i}{dt} &= -\frac{\langle \nabla p \rangle_i}{\rho_i} + \frac{\langle \text{div}(\mathbb{T}^\nu) \rangle_i}{\rho_i} + \vec{g} + \langle \text{div}(\vec{u} \otimes \delta \vec{u}) \rangle_i - \vec{u}_i \langle \text{div}(\delta \vec{u}) \rangle_i \\
\frac{d\vec{r}_i}{dt} &= \vec{u}_i + \delta \vec{u}_i, \quad V_i(t) = m_0_i / \rho_i(t), \quad p = c_0^2 (\rho - \rho_0).
\end{align*}
\]

Formulation in a quasi-lagrangian framework

Advection correction must be done in a consistent framework

Taylor-Green vortex flow

Pressure history measured in the center

$t = 0$, $L/\Delta x = 50$

$\frac{p}{p(t_0)}$

$Re = 1000$

$L/\Delta x = 400$
Advection correction must be done in a consistent framework.

Pressure profile:

\[
\frac{P}{P_0} = \begin{cases} 
1 & \text{Re}=1000, \ tU/L=6.00, \\
2 & \text{Section } y=0.5L 
\end{cases}
\]

Pressure history measured in the center:

\[
\frac{p}{p(t_0)} = \begin{cases} 
1 & \text{Re} = 1000 \\
L/\Delta x =400
\end{cases}
\]
Applications to complex problems

All these enhancements bring clear benefits to practical simulations

Extreme vertical sloshing in an aircraft wing tank

Flow is accelerated up to ~10g and the tank oscillates at a frequency of 6.5 Hz
Applications to complex problems

All these enhancements bring clear benefits to practical simulations
Applications to complex problems

All these enhancements bring clear benefits to practical simulations

In order to accurately reproduce this phenomenon we need:

- Pressure fields free from numerical noise → Diffusive terms
- Accurate vorticity fields and low numerical diffusion → Quasi-Lagrangian scheme
- Turbulence modelling → LES model

Comparison of vertical forces

Comparison of dissipated energy

Some recent developments

A Quasi-Lagrangian scheme with higher-order diffusive terms

Primitive Variable
Riemann Solver (PVRS) in place of
δ-SPH diffusive terms

\[
\begin{aligned}
\frac{d\rho_i}{dt} &= -\rho_i \text{div}(u_i + \delta u_i) + \text{div}(\rho_i \delta u_i) + \Theta_{i,\text{Rie}}^\rho \\
\rho_i \frac{du_i}{dt} &= F_i^p + F_i^\mu + f_i + \text{div}(\rho_i u_i \otimes \delta u_i) + \Theta_{i,\text{Rie}}^u \\
\frac{dx_i}{dt} &= u_i + \delta u_i, \quad V_i(t) = m_i/\rho_i(t), \quad P_i = c_0^2(\rho_i - \rho_0)
\end{aligned}
\]

\[
\Theta_{i,\text{Rie}}^\rho = -\rho_i \sum_i (2u_E - (u_i + u_j)) \cdot \nabla W_{ij} V_j
\]

\[
\Theta_{i,\text{Rie}}^u = -\sum_j [2P_E - (P_i + P_j)] \nabla_i W_{ij} V_j
\]

where \(u_E\) and \(P_E\) are the solutions of the Riemann problem

Some recent developments

A Quasi-Lagrangian scheme with higher-order diffusive terms

\[
\begin{align*}
\frac{d\rho_i}{dt} &= -\rho_i \text{div}(u_i + \delta u_i) + \text{div}(\rho_i \delta u_i) + \Theta_{i,Rie}^\rho \\
\rho_i \frac{du_i}{dt} &= F_i^\rho + F_i^\mu + f_i + \text{div}(\rho_i u_i \otimes \delta u_i) + \Theta_{i,Rie}^u \\
\frac{dx_i}{dt} &= u_i + \delta u_i, \quad V_i(t) = m_i/\rho_i(t), \quad P_i = c_0^2(\rho_i - \rho_0)
\end{align*}
\]

It can be shown that:

\[
\Theta_{i,Rie}^\rho = - \frac{3}{8} R^2 \alpha (\nabla \cdot u) + \frac{1}{c_0 \rho_0} \frac{R^3}{16} \beta \Delta^2 P + \mathcal{O}(R^4)
\]

\[
\Theta_{i,Rie}^u = + \frac{3}{8} R^2 \alpha \nabla (\Delta P) - \rho_0 c_0 \frac{R^3}{120} \gamma \left[ \Delta^2 u + 4 \nabla (\Delta (\nabla \cdot u)) \right] + \mathcal{O}(R^4)
\]
Some recent developments

A Quasi-Lagrangian scheme with higher-order diffusive terms

Primitive Variable Riemann Solver (PVRS) in place of δ-SPH diffusive terms

It can be shown that:

\[
\Theta_{i,Rie}^{\rho} = -\frac{3}{8} \frac{R^2}{\alpha} \Delta (\nabla \cdot u) + \frac{1}{c_0 \rho_0} \frac{R^3}{16} \beta \Delta^2 P + O(R^4)
\]

\[
\Theta_{i,Rie}^{\mu} = +\frac{3}{8} R^2 \alpha \nabla (\Delta P) - \rho_0 \frac{R^3}{120} \gamma \left[ \Delta^2 u + 4 \nabla (\Delta (\nabla \cdot u)) \right] + O(R^4)
\]

bi-laplacian as in δ-SPH!
(see also Green et al. 2019)
Some recent developments

A Quasi-Lagrangian scheme with higher-order diffusive terms

Primitive Variable Riemann Solver (PVRS) in place of δ-SPH diffusive terms

It can be shown that:

\[
\begin{aligned}
\frac{d\rho_i}{dt} &= -\rho_i \text{div}(u_i + \delta u_i) + \text{div}(\rho_i \delta u_i) + \Theta^\rho_{i,\text{Rie}} \\
\rho_i \frac{du_i}{dt} &= F_i^\rho + F_i^\mu + f_i + \text{div}(\rho_i u_i \otimes \delta u_i) + \Theta^\mu_{i,\text{Rie}} \\
\frac{dx_i}{dt} &= u_i + \delta u_i, \quad V_i(t) = m_i/\rho_i(t), \quad P_i = c_0^2(\rho_i - \rho_0)
\end{aligned}
\]

\[
\Theta^\rho_{i,\text{Rie}} = -\frac{3}{8} \frac{R^2}{\alpha} \Delta (\nabla \cdot u) + \frac{1}{c_0 \rho_0} \frac{R^3}{16} \beta \Delta^2 P + \mathcal{O}(R^4)
\]

diffusive term in momentum eq. is of higher order than artificial viscosity

\[
\Theta^\mu_{i,\text{Rie}} = +\frac{3}{8} \frac{R^2}{\alpha} \nabla (\Delta P) - \rho_0 \frac{R^3}{120} \gamma \left[ \Delta^2 u + 4 \nabla (\Delta (\nabla \cdot u)) \right] + \mathcal{O}(R^4)
\]
Some recent developments

A Quasi-Lagrangian scheme with higher-order diffusive terms

(a) Pressure field compared to analytic solution

\( \frac{L}{\Delta x} = 800 \), \( \frac{tU}{L} = 5 \)

Observed convergence order between 2 and 3!

Acoustic waves sometimes are an unwanted noise on the solution.
Acoustic waves can be unwanted noise on the solution.
Acoustic waves can be unwanted noise on the solution.
The $\delta^+$-SPH with the acoustic damper term

\[
\begin{align*}
\frac{d\rho_i}{dt} &= -\rho_i \sum_j (u_{ji} + \delta u_{ji}) \cdot \nabla_i W_{ij} V_j \\
&\quad + \sum_j (\rho_j \delta u_j + \rho_i \delta u_i) \cdot \nabla_i W_{ij} V_j + D_i^p \\
\rho_i \frac{d\mathbf{u}_i}{dt} &= \mathbf{F}_i^p + \mathbf{F}_i^v + \mathbf{F}_i^{ad} + \rho_i \mathbf{g} \\
&\quad + \sum_j (\rho_j \mathbf{u}_j \otimes \delta \mathbf{u}_j + \rho_i \mathbf{u}_i \otimes \delta \mathbf{u}_i) \cdot \nabla_i W_{ij} V_j \\
\frac{dr_i}{dt} &= \mathbf{u}_i + \delta \mathbf{u}_i, \quad V_i = m_i / \rho_i, \quad p = c_0^2 (\rho - \rho_0).
\end{align*}
\]
Diffusive terms for low-frequency acoustic noise

The $\delta^+$-SPH with the acoustic damper term

\[ F_{i}^{ad} = \alpha_2 \rho_0 c_0 h \sum_{j} \left( \dot{c}_j + \dot{c}_i \right) \nabla_i W_{ij} V_j \approx \xi \nabla \left( \nabla \cdot \mathbf{u} \right) \]

\[ \dot{c}_k = -\frac{\rho_k}{\rho_k} = \sum_{l} u_{lk} \cdot \nabla_k W_{kl} V_l \approx \nabla \cdot \mathbf{u} \]

\[ \xi = \alpha_2 \rho_0 c_0 h \] is a free bulk coefficient where $\alpha_2$ is a dimensionless coefficient.

Similar to the artificial viscosity of Monaghan & Gingold, $\xi$ decreases as $h$ goes to zero

Diffusive terms for low-frequency acoustic noise

Dam-break flow against a rectangular obstacle

without acoustic damper  
with acoustic damper ($\alpha_2 = 1$)
Wrapping up

- Particle methods, such as SPH, are more prone to particle-scale noise with respect to grid-based schemes because of additional degrees of freedom.
- Diffusive terms are a straightforward way to alleviate this issue.
- One can follow different paths to derive diffusive terms in the continuity equations, some of those are not merely numerical expedients but rely on physical bases (e.g. $\delta$-LES).
- They are even more effective in combination with particle shifting techniques (Quasi-Lagrangian) as the latter acts on the source of the particle-scale noise.
- Specific diffusive terms can be also conceived to address acoustic noise stemming from the weak-compressibility assumption.
- All these ingredients allow for tackling complex and challenging problems and ongoing research continuously increases the capabilities of the SPH scheme.