Modeling viscous forces in SPH

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Outline

1. Viscous forces in Fluid Mechanics
2. Artificial viscosity in SPH
3. Viscous forces and SPH derivatives: the Español&Revenga formula.
4. Common SPH viscous terms.
5. SPH, viscosity and FSs.
6. SPH and bulk viscosity.

Viscous forces in Fluid Mechanics

For the sake of simplicity and practical interest, let's restrict ourselves to Newtonian fluids.

Viscous forces in Fluid Mechanics

Newtonian Viscous term

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla p + \rho\, \mathbf{a}^{v}(\mathbf{r}) + \rho \mathbf{g}$$

Viscous acceleration:

$$\rho\, \mathbf{a}^{v}(\mathbf{r}) = \mu \nabla^2 \mathbf{v}(\mathbf{r}) + (\mu + \lambda) \nabla (\nabla \cdot \mathbf{v})(\mathbf{r})$$

μ	shear viscosity
λ	second viscosity

Second order derivatives are necessary to model viscous terms

Viscous forces in Fluid Mechanics

1. Viscous forces are dominant in boundary layers and in turbulent wakes.
2. They are important to describe friction drag, separation, turbulence, added damping in floating structures, etc..

Viscous forces in Fluid Mechanics

1. The question is: Are boundary layers, separation, etc. important in the problems we are interested in?

If answer is YES, SPH usually struggles with these problems and is competitive only if distorted/fragmented FSs are involved.

Viscous forces in Fluid Mechanics

1. The question is: Are boundary layers, separation, etc. important in the problems we are interested in?

If answer is NO, then at least viscous forces are diffusive and help stabilizing numerical schemes (artificial viscosity)

Artificial viscosity

SPH was created in 1977 by Gingold&Monaghan and (independently) by Lucy, targeting gas dynamics problems (Euler eqs).

Viscous effects were not important.

Stabilizing the time integration scheme to resolve shocks appearing as solutions of Euler eqs. was however necessary.

Artificial viscosity

Gingold&Monaghan (1983) developed an artificial viscosity term that conserved linear and angular momentum exactly (important for shocks), that vanished with increasing resolution, and that worked well

$$\frac{\mathrm{d}\mathbf{v}_a}{\mathrm{d}t} = -\sum_{b\in\mathcal{N}_a} m_b \left(\frac{P_a}{\rho_a^2} + \frac{P_b}{\rho_b^2}\right) \nabla_a W_{ab} + \mathbf{\Pi}_a$$

$$\boxed{\mathbf{\Pi}_a = -\alpha h \sum_b \frac{m_b}{\bar{\rho}_{ab}} c_s \frac{\mathbf{v}_{ab}\cdot\mathbf{r}_{ab}}{r_{ab}^2 + \xi^2} \nabla_a W_{ab}}$$

α artificial viscosity coeff.

c_s numerical sound speed

h smoothing length

Artificial viscosity

This term can be written as an integral and taken back to the continuum, leading to:

$$\rho\, \mathbf{a}^{MG}(\mathbf{r}) \;=\; \mu\, \nabla^2 \mathbf{v}(\mathbf{r}) \;+\; 2\,\mu\, \nabla\, (\nabla \cdot \mathbf{v})(\mathbf{r})$$

with:

$$\boxed{\frac{\mu}{\rho} = \frac{\alpha\, h\, c_s}{2(n+2)}} \qquad n = \text{dimensionality}$$

Colagrossi et al. (2017)

$$\mathbf{\Pi}_a = -\alpha h \sum_b \frac{m_b}{\bar{\rho}_{ab}} c_s \frac{\mathbf{v}_{ab} \cdot \mathbf{r}_{ab}}{r_{ab}^2 + \xi^2} \nabla_a W_{ab}$$

Artificial viscosity

Therefore, when artificial viscosity is used, it is equivalent to having a Newtonian viscous term.

An equivalent Reynolds number can be therefore estimated:

$$Re = \frac{\rho\, U\, L}{\mu} \approx \{2D\} \approx \frac{8\, U\, L}{\alpha\, h\, c_s}$$

Artificial viscosity

Let’s make an example: dambreak

Artificial viscosity

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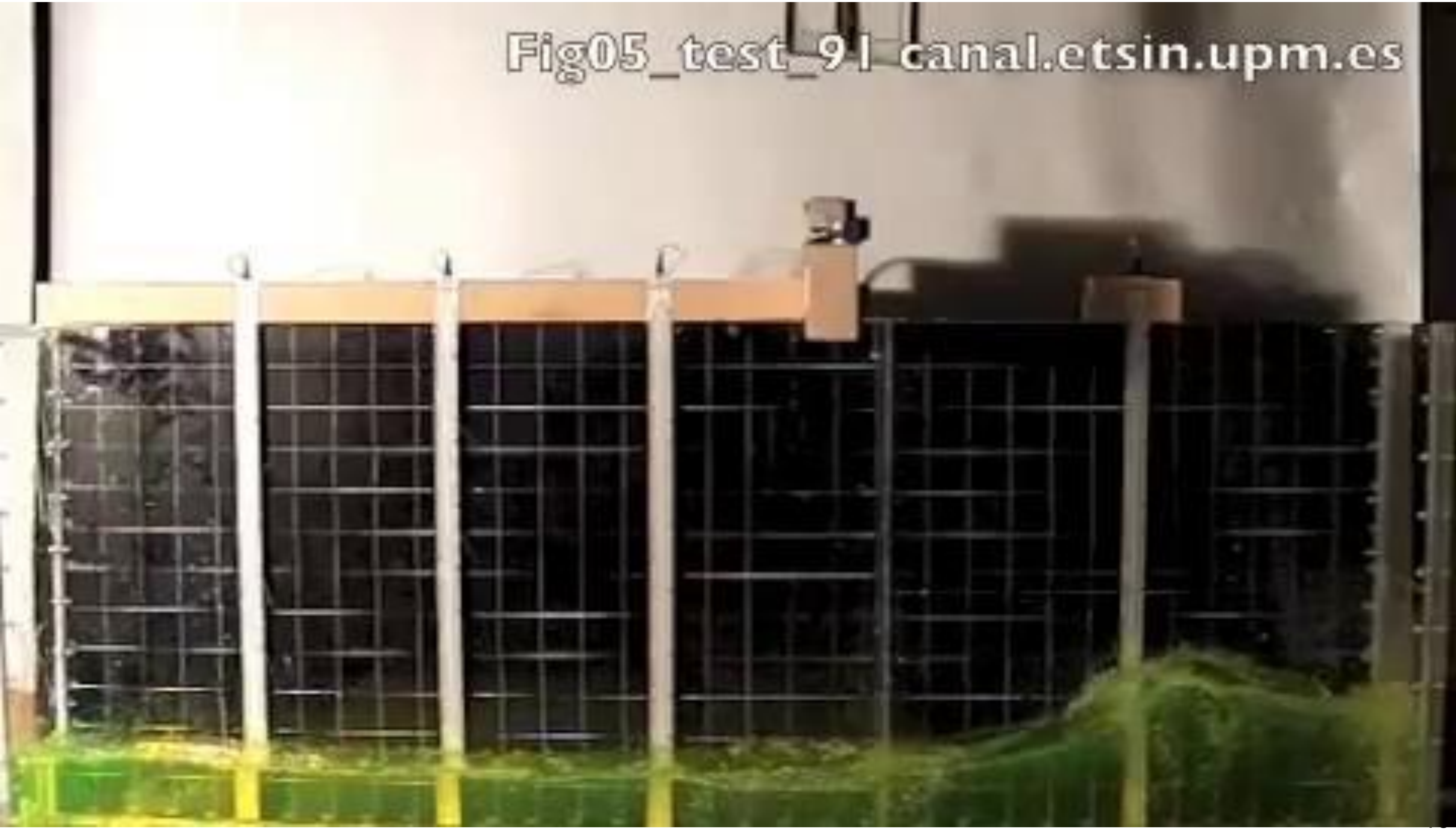

Artificial viscosity

Let's make an example: dambreak

$$Re_{physical} = \frac{\rho\, U\, L}{\mu} \approx \frac{10^3 \cdot 1 \cdot 1}{10^{-3}} \approx 10^6$$

$$Re_{SPH} \approx \frac{8\, U\, L}{\alpha\, h\, c_s} \approx \{2 \cdot 10^5 \text{ particles}\} \approx \frac{8 \cdot 1 \cdot 1}{0.02 \cdot 0.0025 \cdot 10} = 16000$$

There is a factor of 62.5 between actual and SPH Re. Therefore we should have 62.5^2 = 800 million particles to be able to get the actual Re.

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Viscous forces and SPH derivatives

Why does nobody use second order derivatives of the kernel for modeling viscous terms?

$$\left(\frac{d^2 f}{dx^2}\right)_a = \sum_b f_b \left.\frac{d^2 W(x - x_b, h)}{dx^2}\right|_{x=x_a} V_b$$

1. The expression is very sensitive to particle disorder.
2. Since the second derivative changes sign, the diffusion processes (either viscous or thermal) may not have the right direction (transfer of mechanical energy from faster to slower particles + idem with heat)

Viscous forces and SPH derivatives

How can be second derivatives computed without resorting to kernel's second derivative?

$$\boxed{\nabla_a W(r_{ab}) = -\mathbf{r}_{ab} F(r_{ab})}$$

Notation

$$\boxed{\Delta x' = x' - x}$$

$$I(\mathbf{r}) := \int (f(x') - f(x)) F(|\Delta x'|)\, dx'$$

Taylor exp.

$$\approx \int \left(f(x) + \frac{df}{dx}\Delta x' + \frac{d^2 f}{dx^2}\frac{\Delta x'^2}{2} - f(x) \right) F(|\Delta x'|)\, dx'$$

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$$= \frac{1}{2}\frac{d^2 f}{dx^2} \int \Delta x'^2 F(|\Delta x'|)\, dx' = \boxed{\frac{1}{2}\frac{d^2 f}{dx^2}}$$

Macia et al, PTP2011

Viscous forces and SPH derivatives

How does this read at the particle level?

$$\left.\frac{d^2 f}{dx^2}\right|_{x=x_a} \approx 2 \int (f(x') - f(x)) F(|\Delta x'|)\, dx'$$

$$\approx 2 \sum_b \frac{f_b - f_a}{x_b - x_a} \left.\frac{dW(x - x_b, h)}{dx}\right|_{x=x_a} V_b$$

Viscous forces and SPH derivatives

Español and Revenga (2003) generalized this to any derivatives, but Morris et al (1997) had already used this formula to model the velocity Laplacian and Cleary (1997) for heat transfer.

$$\frac{\mathrm{d}\mathbf{v}_a}{\mathrm{d}t} = -\sum_{b\in\mathcal{N}_a} m_b \left(\frac{P_a}{\rho_a^2} + \frac{P_b}{\rho_b^2}\right) \nabla_a W_{ab} + \mathbf{\Pi}_a$$

$$\boxed{\mathbf{\Pi}_a^{\mathrm{MVT}} = \sum_b \frac{m_b}{\rho_a \rho_b} (\mu_a + \mu_b) \mathbf{v}_{ab} \frac{\mathbf{r}_{ab} \cdot \nabla_a W_{ab}}{r_{ab}^2}}$$

Most common viscous terms

Morris et al (1997)

$$\frac{d\mathbf{v}_a}{dt} = -\sum_{b \in \mathcal{N}_a} m_b \left(\frac{P_a}{\rho_a^2} + \frac{P_b}{\rho_b^2} \right) \nabla_a W_{ab} + \mathbf{\Pi}_a$$

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$$\rho\, \mathbf{a}^{Morris}(\mathbf{r}) = \mu\, \nabla^2 \mathbf{v}(\mathbf{r})$$

Models incompressible flow viscous term.
Does not conserve angular momentum exactly.

Most common viscous terms

Monaghan&Gingold (1983)

$$\frac{d\mathbf{v}_a}{dt} = -\sum_{b \in \mathcal{N}_a} m_b \left(\frac{P_a}{\rho_a^2} + \frac{P_b}{\rho_b^2} \right) \nabla_a W_{ab} + \mathbf{\Pi}_a$$

$$\Pi^{MGVT} = -\frac{2(n+2)\mu}{\rho_a} 2 \sum_b \frac{\mathbf{v}_{ab} \cdot \mathbf{r}_{ab}}{r_{ab}^2} \boxed{\nabla_a W_{ab}} V_b$$

$$\rho\, \mathbf{a}^{MG}(\mathbf{r}) = \mu \nabla^2 \mathbf{v}(\mathbf{r}) + 2\mu \nabla (\nabla \cdot \mathbf{v})(\mathbf{r})$$

Angular momentum is exactly conserved, but an additional compressible viscosity term appears.

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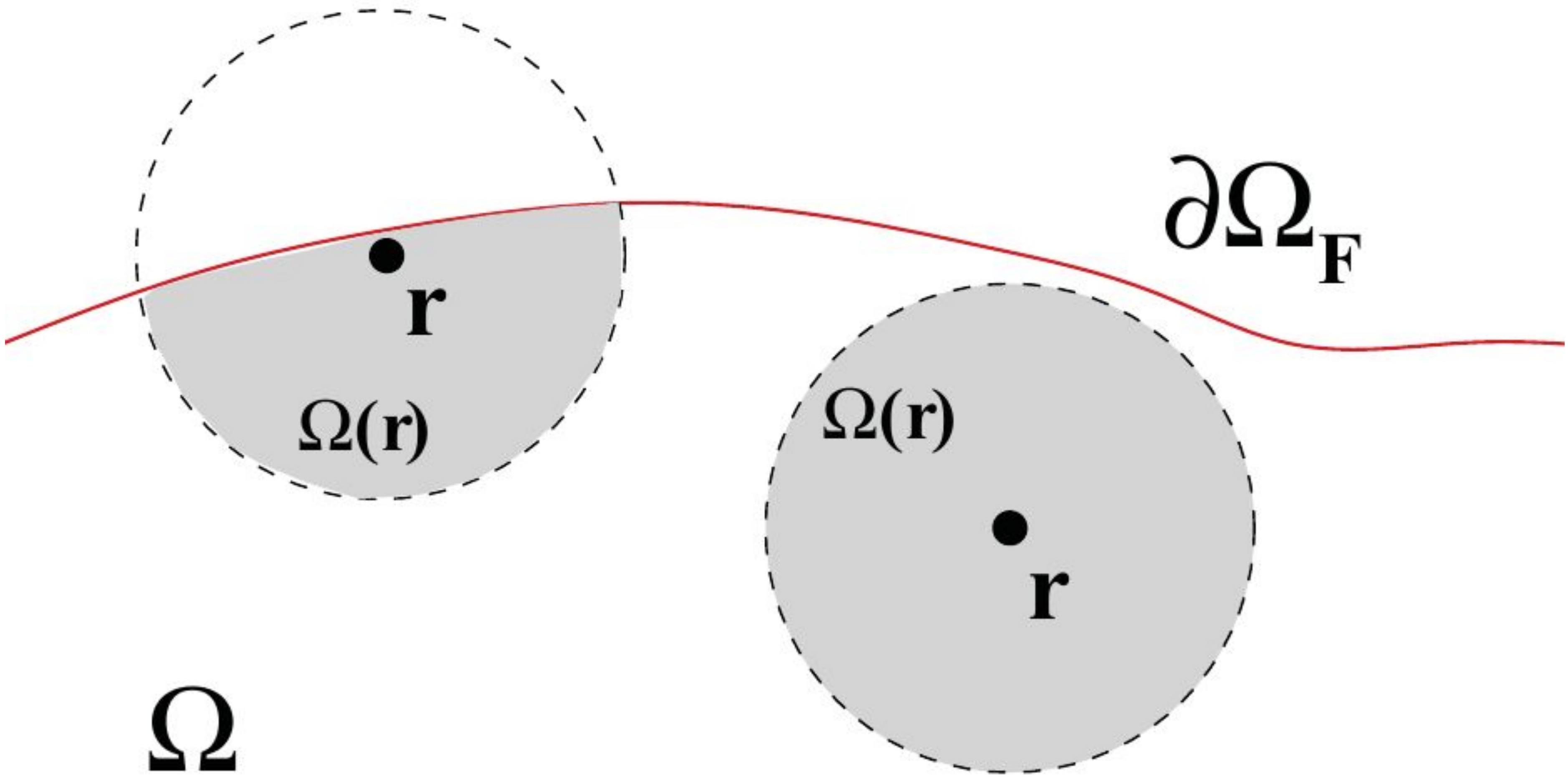
SPH, viscosity and FSs.

SPH is a competitive method for energetic free-surface flows. The following question arises naturally:

How do these viscous terms perform with truncated domains, and, more specifically, in the presence of a free surface?

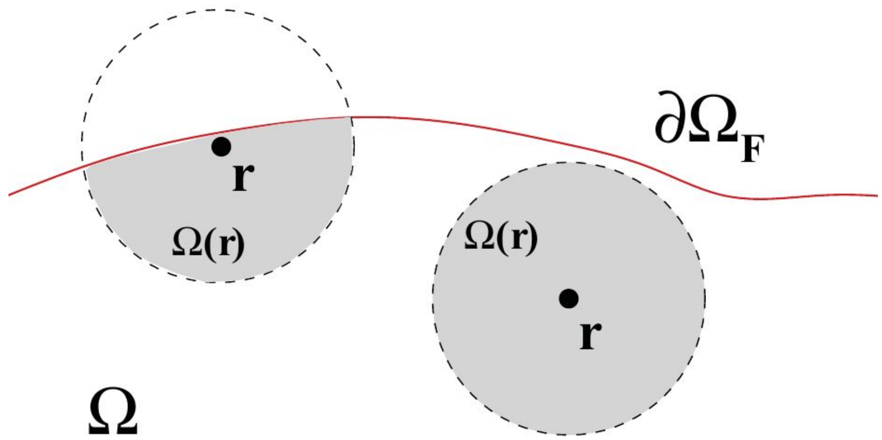
Colagrossi et al., 2011

SPH, viscosity and FSs.



Kernel support and free-surface

SPH, viscosity and FSs.



$$I(\mathbf{r}) := \int (f(x') - f(x)) F(|\Delta x'|)\, dx'$$

Taylor exp.

$$\approx \int \left(\cancel{f(x)} + \cancel{\frac{df}{dx}\Delta x'} + \frac{d^2 f}{dx^2}\frac{\Delta x'^2}{2} - \cancel{f(x)} \right) F(|\Delta x'|)\, dx'$$

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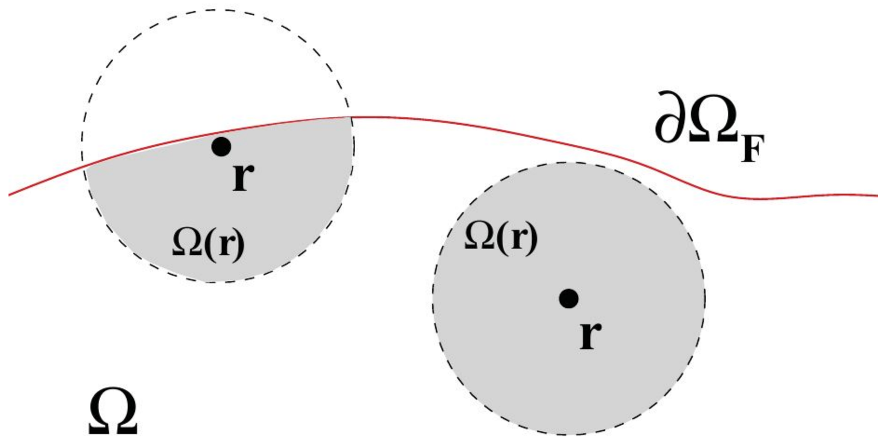

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Kernel support and free-surface

SPH, viscosity and FSs.

$$\rho\, \mathbf{a}^{MG}(\mathbf{r}) \cdot \mathbf{n} = \mu \frac{C}{h} (\nabla \cdot \mathbf{v} + \mathbf{n} \cdot \nabla \mathbf{v} \cdot \mathbf{n})$$

SPH, viscosity and FSs.

$$\rho\,\mathbf{a}^{MG}(\mathbf{r})\cdot\mathbf{n} = \mu\frac{C}{h}(\nabla\cdot\mathbf{v} + \mathbf{n}\cdot\nabla\mathbf{v}\cdot\mathbf{n})$$

the force is singular!

SPH, viscosity and FSs.

$$\rho\, \mathbf{a}^{MG}(\mathbf{r}) \cdot \mathbf{n} \;=\; \mu \frac{C}{h} \left(\cancel{\nabla \mathbf{v}} + \mathbf{n} \cdot \nabla \mathbf{v} \cdot \mathbf{n} \right)$$

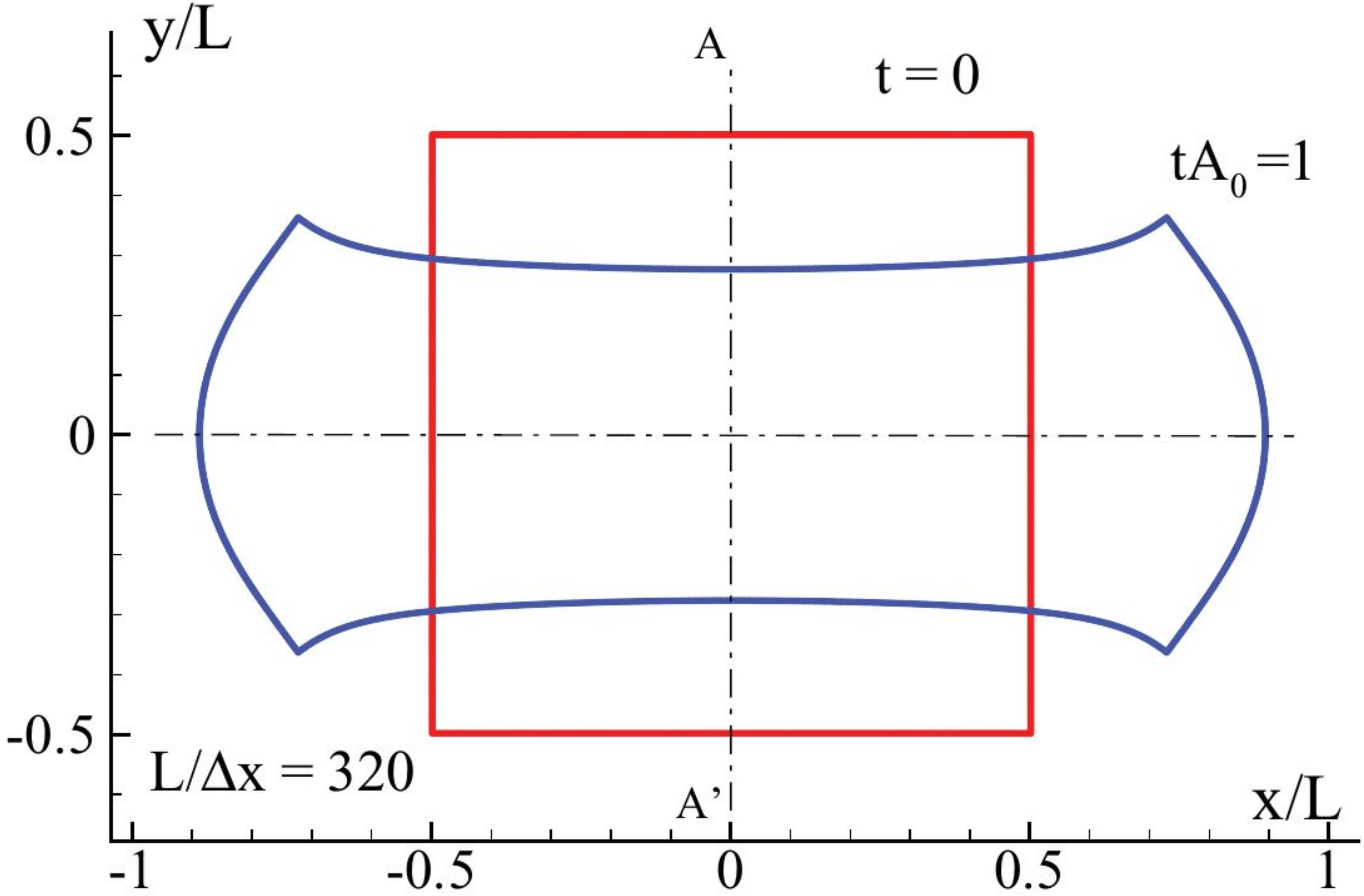
the force is singular!

A similar thing happens with Morris et al. term

$$\rho\, \mathbf{a}^{Mo}(\mathbf{r}) \cdot \mathbf{n} \;=\; \mu \frac{C'}{h} \left(\mathbf{n} \cdot \nabla \mathbf{v} \cdot \mathbf{n} \right)$$


What are you saying guys ??!!*

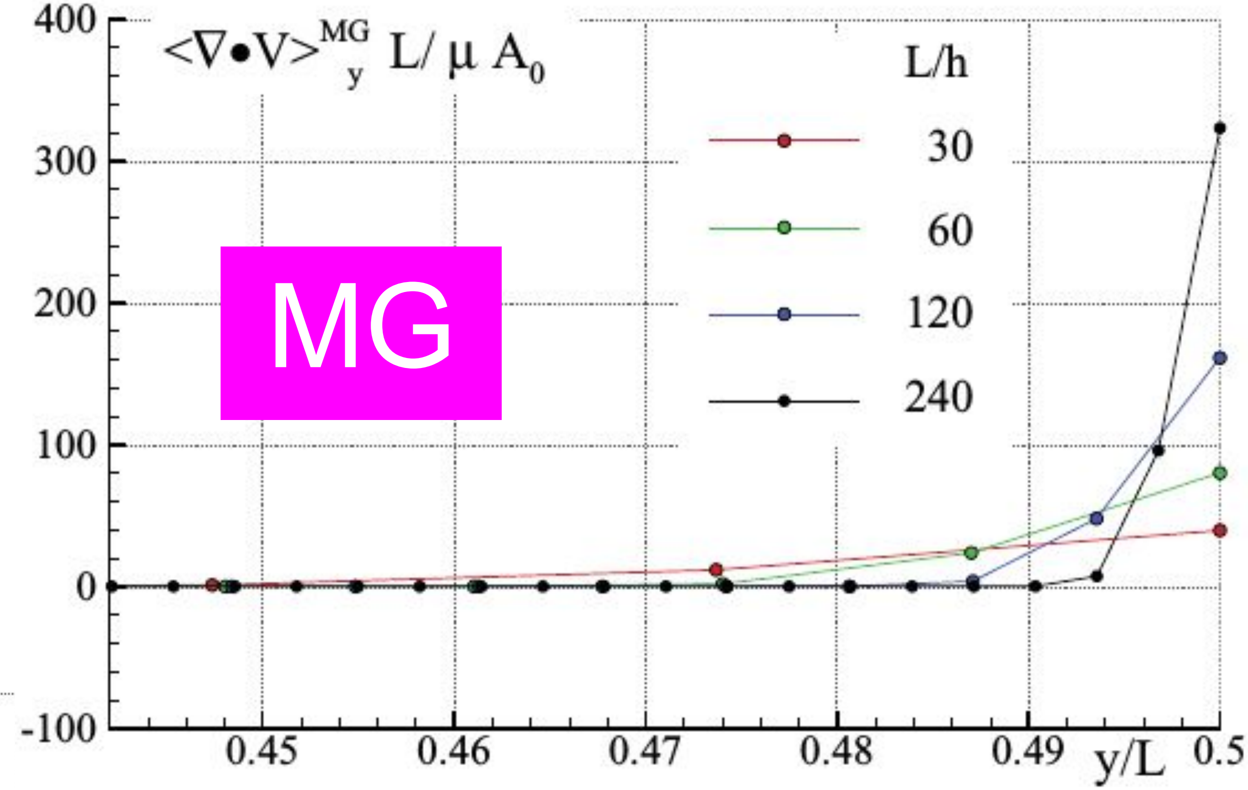
SPH, viscosity and FSs.

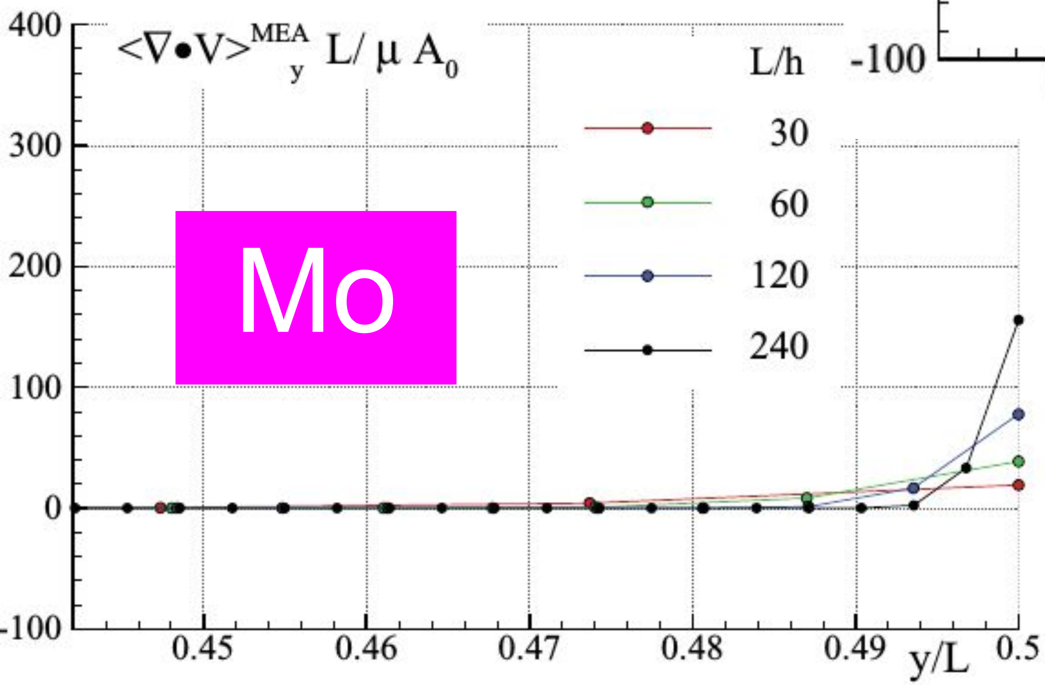


$$\begin{cases} u_0(x,y) = A_0 x \\ v_0(x,y) = -A_0 y \end{cases}$$

SPH, viscosity and FSs.

AA' line





SPH, viscosity and FSs.

But if we look at total dissipation.

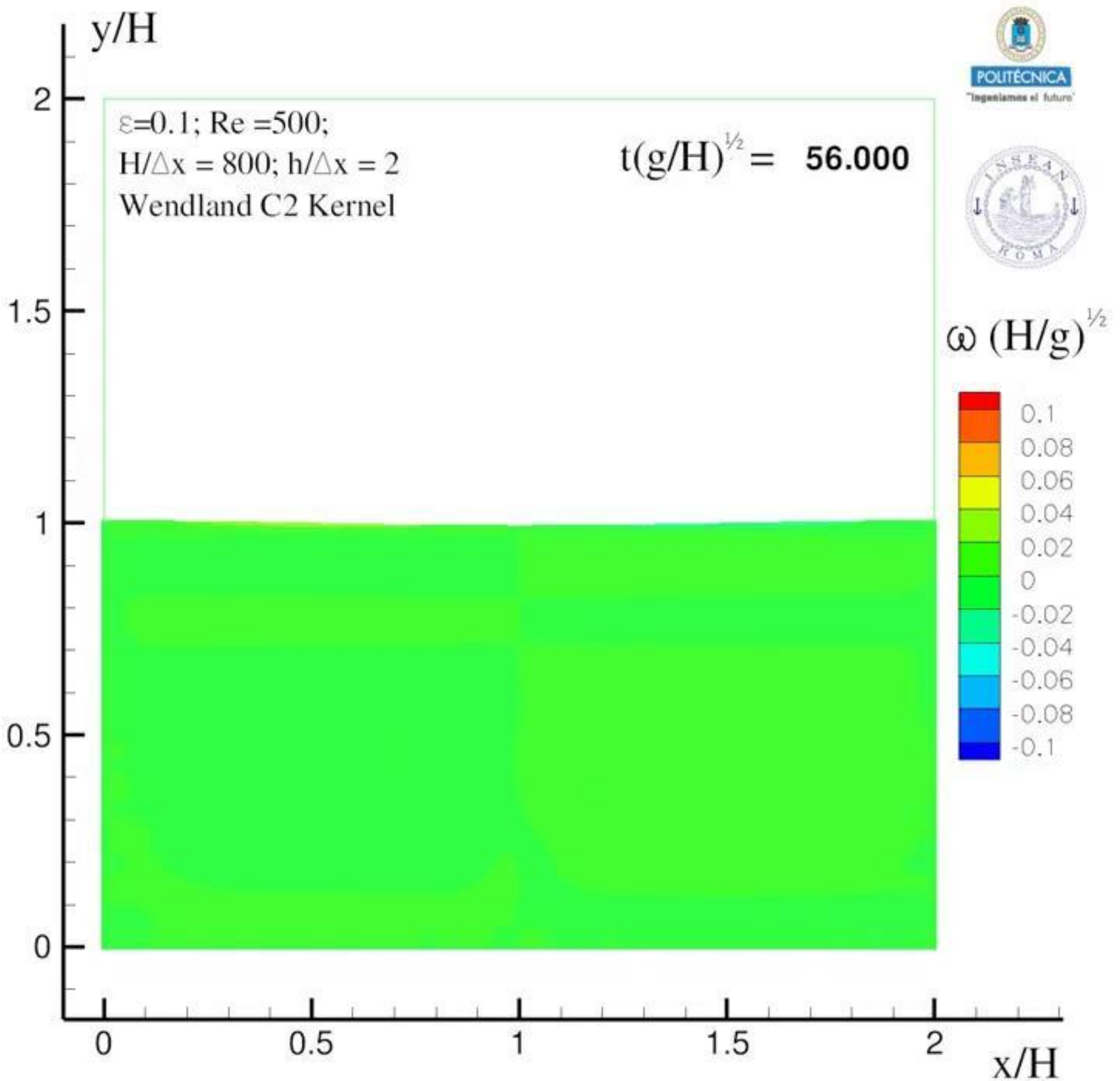
$$\int_\Omega \mathbf{v} \cdot < \nabla \cdot \mathbb{V} >^{MG} dV = \boxed{-2\mu \int_\Omega \mathbb{D} : \mathbb{D} \, dV} + O(h),$$

$$\int_\Omega \mathbf{v} \cdot < \nabla \cdot \mathbb{V} >^{Mo} dV = -\mu \int_\Omega \|\nabla \boldsymbol{u}\|^2 dV + O(h).$$

Morris formula in the presence of a FS does not converge to the exact dissipation

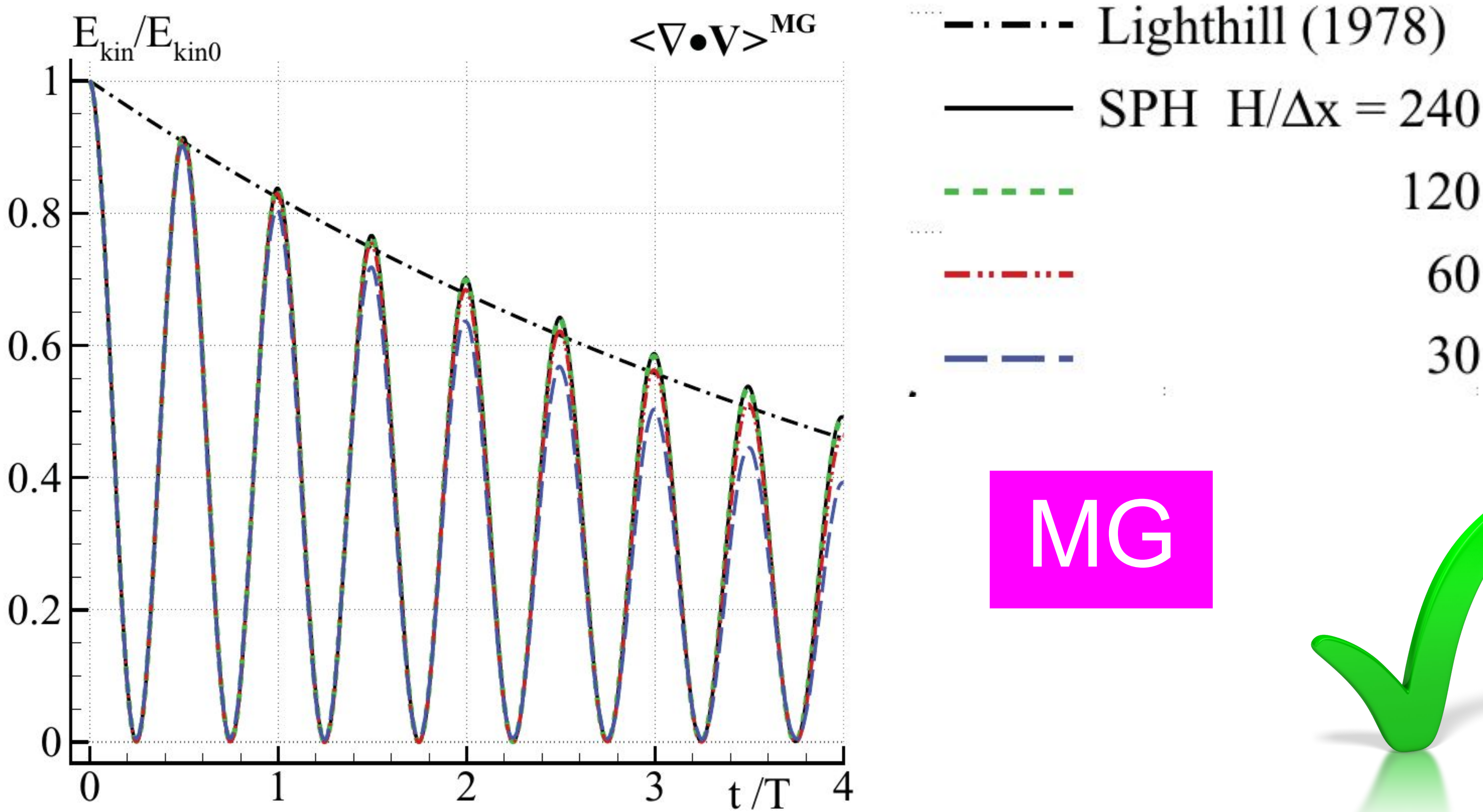
SPH, viscosity and FSs.

Standing wave decay test case



SPH, viscosity and FSs.

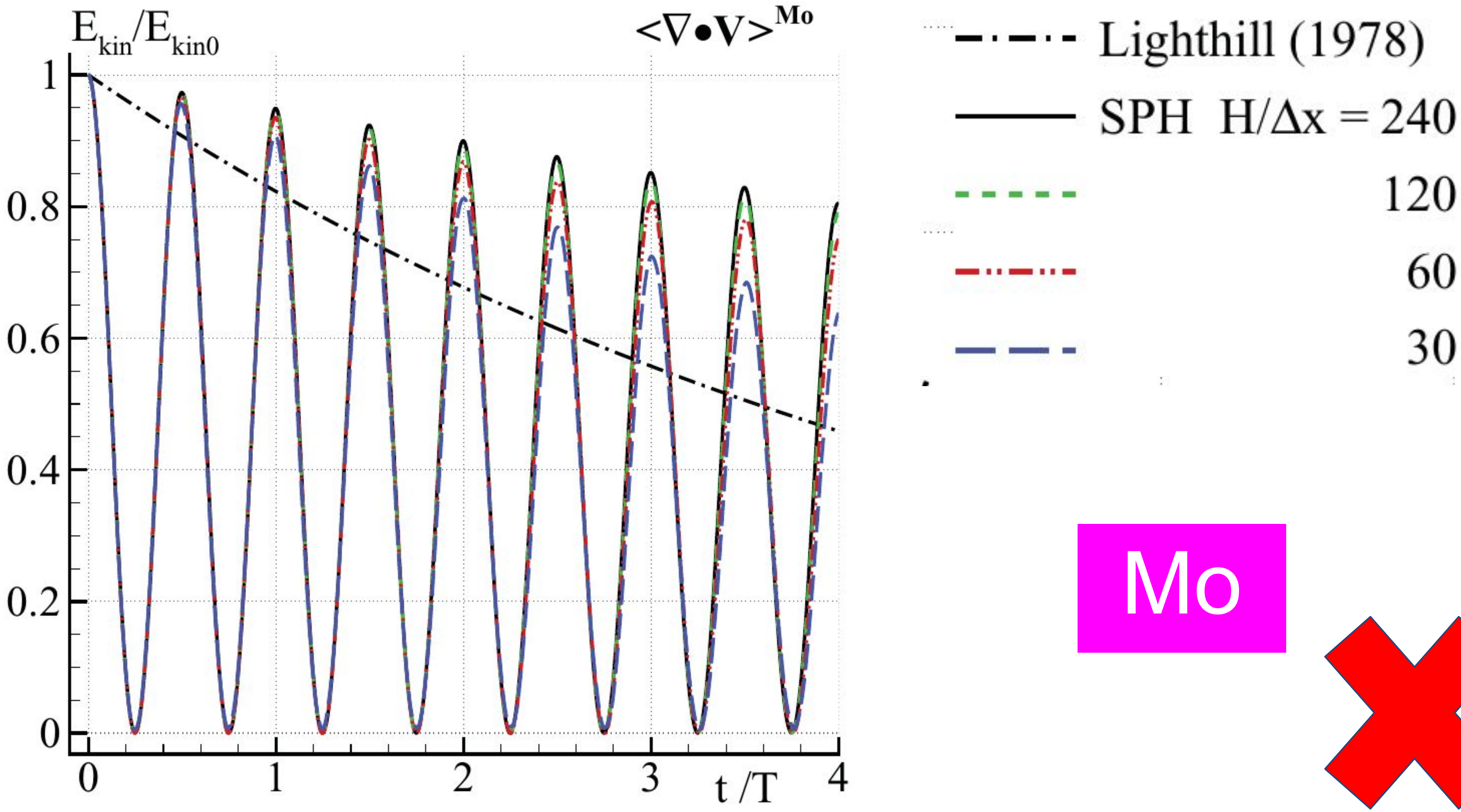
Standing wave decay test case



MG

SPH, viscosity and FSs.

Standing wave decay test case



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6. **SPH and bulk viscosity.**

(Avalos, J.B. et al, 2020)

Motivation

1. If SPH viscous terms based on central forces are considered (no rotational mode excited and angular momentum conserved),
2. if linear momentum is conserved
3. if these terms are in agreement with 2nd law,

then the viscous term is Monaghan&Gingold's:

$$\Pi^{MGVT} = -\frac{2(n+2)\mu}{\rho_a} 2 \sum_b \frac{\mathbf{v}_{ab} \cdot \mathbf{r}_{ab}}{r_{ab}^2} \boxed{\nabla_a W_{ab}} V_b$$

Colagrossi et al. PRE (2017):

Motivation

by transforming back summations to integrals, it becomes:

$$\rho\, \mathbf{a}^{MG}(\mathbf{r}) \;=\; \mu\, \nabla^2 \mathbf{v}(\mathbf{r}) \;+\; 2\,\mu\, \nabla\, (\nabla \cdot \mathbf{v})(\mathbf{r})$$

If one compares it with the Newtonian one:

$$\rho\, \mathbf{a}^{v}(\mathbf{r}) \;=\; \mu\, \nabla^2 \mathbf{v}(\mathbf{r}) \;+\; (\mu + \lambda)\, \nabla\, (\nabla \cdot \mathbf{v})(\mathbf{r})$$

one gets: $\boldsymbol{\lambda = \mu}$

$$\kappa := \lambda + \frac{2\,\mu}{n}$$

$$\kappa = \frac{5}{3}\mu$$ 3D

second viscosity

dimensionality

which implies that the MG viscous term incorporates a bulk viscosity, which is not a free parameter, and is proportional and of the same order as the shear viscosity.

Bulk viscosity is the fluid property connected with the compressibility dissipation.

SPH prevalent formulation is Weakly-compressible, and therefore bulk viscosity is interesting

$$\Phi_D = 2\mu \int_\Omega \mathbb{S} : \mathbb{S}\, dV + \kappa \int_\Omega (\nabla \cdot \boldsymbol{u})^2\, dV.$$

Objective of our study was:

Devise a compressible viscosity term in SPH, which together with MG term allows to model a Newtonian Viscous term

$$\rho \mathbf{g} = \mu \nabla^2 \boldsymbol{u} + (\lambda + \mu) \nabla (\nabla \cdot \boldsymbol{u}),$$

Such term must be independent of the shear viscosity and must conserve linear and angular momentum.

Applications of this term would be in flows with moderate or high Mach (compressible flows).

Method / From discrete to continuous:

1. Devise a discrete dissipation function which depends on local volumetric variations.
2. Use Lagrangian formulation to obtain motion equations.
3. Verification

From discrete to continuous

$$V_i \equiv \frac{1}{c_i} = \frac{1}{\sum_j W(r_{ij})}.$$

$$\frac{dc_i}{dt} = \sum_j W'(r_{ij})\frac{dr_{ij}}{dt} = -\sum_{j\neq i} F(r_{ij})\, r_{ij}\, \mathbf{e}_{ij}\cdot\mathbf{u}_{ij}.$$

$$\Phi_D[\mathbf{u}_i] = \frac{1}{2}\sum_{i,j>i} \eta F(r_{ij}) r_{ij} (\mathbf{e_{ij}}\cdot\mathbf{u}_{ij})^2\, V_i V_j + \frac{1}{2}\sum_i \zeta V_i^3\, \dot{c}_i^2$$

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\mathbf{r}}_i}\right) - \frac{\partial\mathcal{L}}{\partial\mathbf{r}_i} = -\frac{\partial}{\partial\mathbf{u}_i}\Phi_D,$$

2. From discrete to continuous (2/2)

$$\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{r}}_i}\right) - \frac{\partial \mathcal{L}}{\partial \mathbf{r}_i} = -\frac{\partial}{\partial \mathbf{u}_i}\Phi_D,}$$

$$\mathbf{f}_i = -\mu\,(n+2)\sum_j F(r_{ij})\,(\boldsymbol{u}_{ij}\cdot\boldsymbol{e}_{ij})\,\boldsymbol{e}_{ij}\,V_i\,V_j + \lambda^{SPH}\sum_j\left(\frac{\dot{\rho}_i}{\rho_i}+\frac{\dot{\rho}_j}{\rho_j}\right)F(r_{ij})\,\boldsymbol{e}_{ij}\,V_i\,V_j.$$

forcing this to be a Newtonian term, one obtains:

$$\boxed{\lambda^{SPH} = \kappa - \left(1+\frac{2}{n}\right)\mu,}$$

Verification. 2D. Unidirectional flow

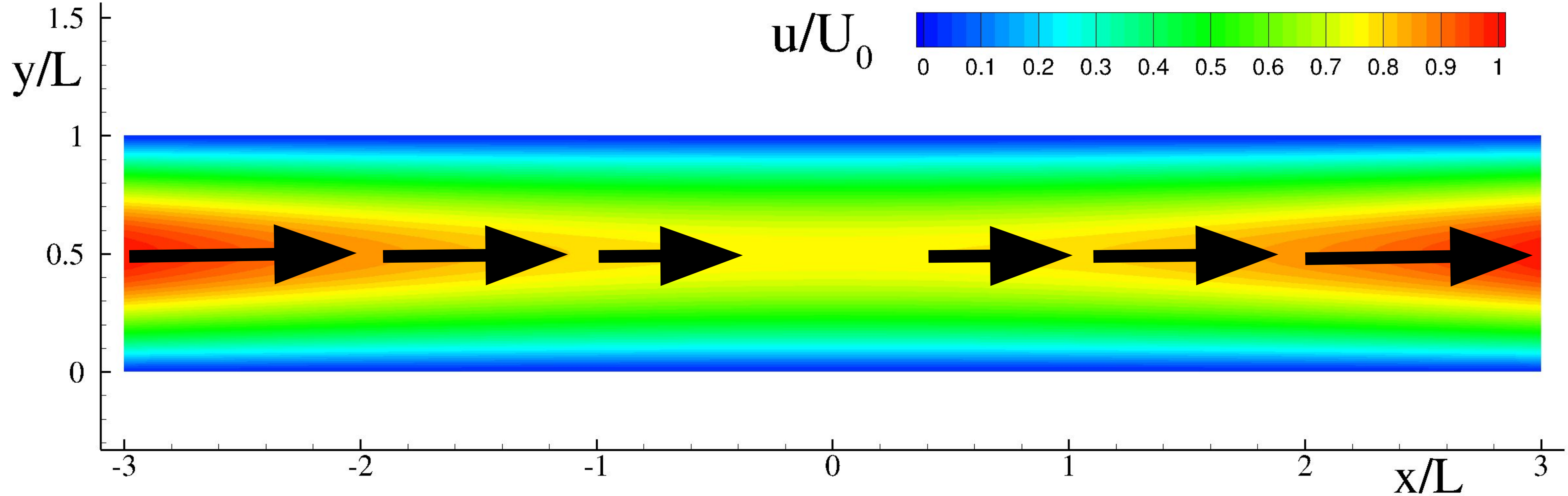
Time decay of an decelerating-accelerating pipe

$$p = c_0^2(\rho - \rho_0) + p_b$$

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ u_t + u u_x = -c_0^2 \dfrac{\rho_x}{\rho} + f + (\lambda' + \nu) u_{xx} + \nu (u_{yy} + u_{xx}) \end{cases}$$

$\lambda' = \lambda/\rho$ and $\nu = \mu/\rho$

f is an external made-up body force that forces this type of flow, leading to compression-expansion dynamics



Verification. 2D

$$f = b\left[\frac{U_0^2}{2}\frac{\sinh(2bx)}{\cosh(bx_0)^2}\sin(ky)^2\exp(-2\nu\alpha^2k^2t) - c_0^2\tanh(bx)\right]$$

$$\alpha = \sqrt{\frac{\sigma+2}{\sigma+3}}, \qquad b = \frac{k}{\sqrt{\sigma+2}\,\sqrt{\sigma+3}}, \qquad \sigma = \frac{\lambda}{\mu}.$$

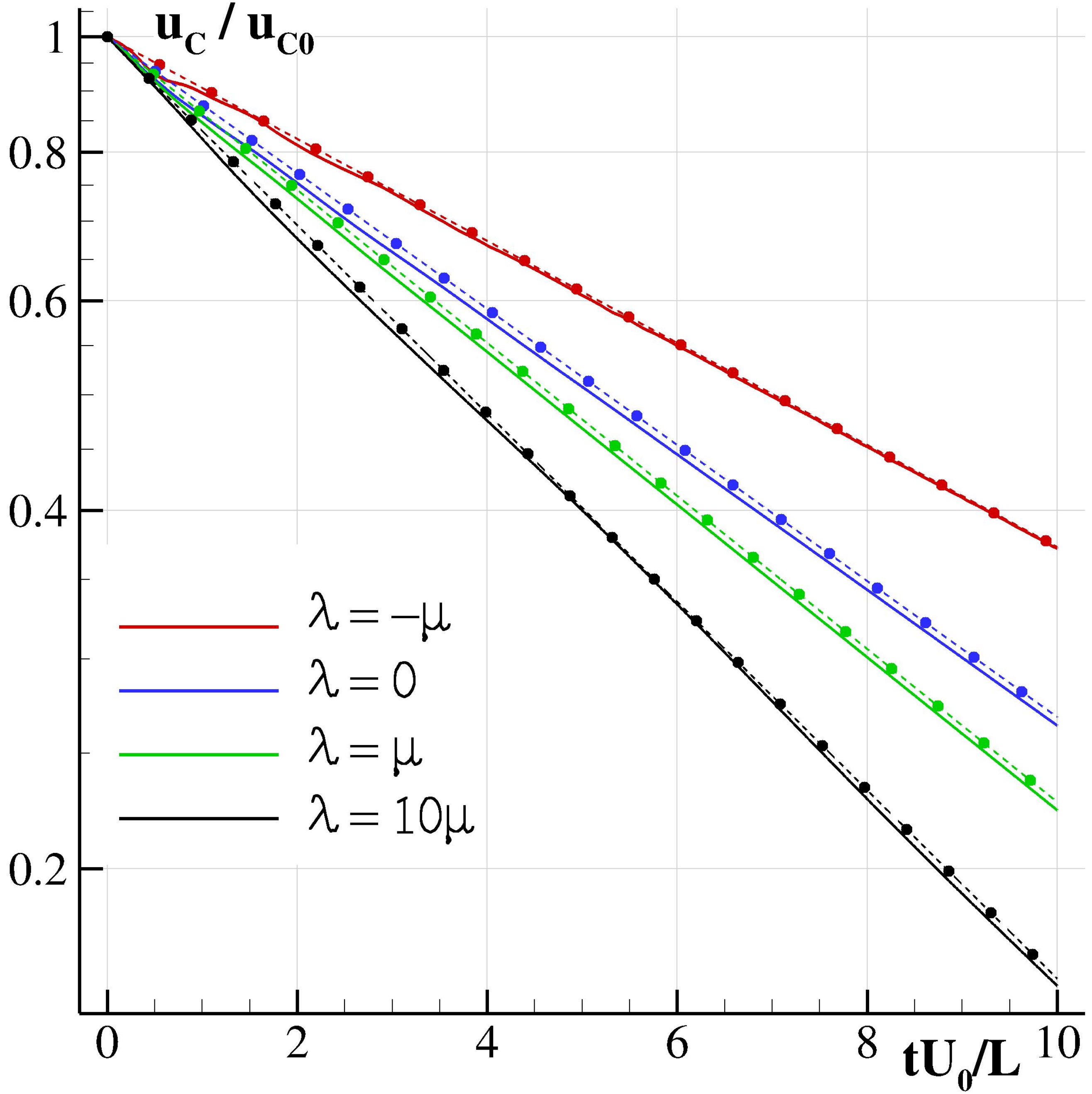
$k = \pi/L$ (L is the wavelength in the y direction)

Splitting-merging necessary (Sun et al. 2019)

An analytical solution for the velocity can be obtained, and thus the kinetic energy

$$u(t,x,y) = U_0\frac{\cosh(bx)}{\cosh(bx_0)}\sin(ky)\exp\left(-\nu\alpha^2k^2t\right)$$

Verification. 2D

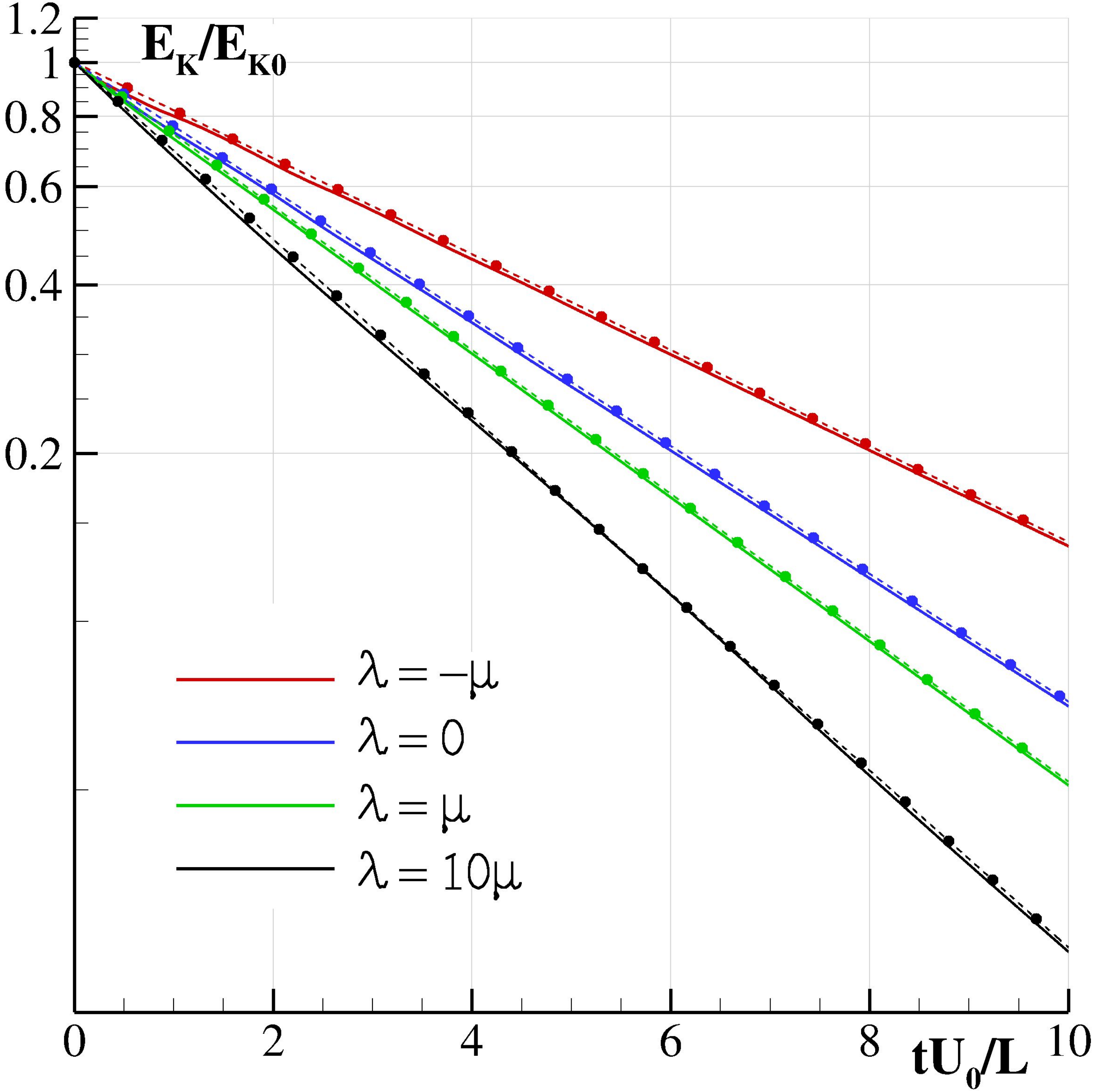


Re=50;

U0/c0=Ma=0.5

decay is faster with increasing λ

Verification. 2D



Outline / Summary
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Take home idea:
Viscous forces may not be crucial in SPH, but I think you will feel more confident using the method if you are aware what’s going on with dissipation and with time integration stabilization techniques.

Transversal topics for another occasion

1. Artificial viscosity vs Riemann solvers
2. Delta-SPH and dissipation.
3. Coupling of SPH with FVM solvers, better suited to resolve viscous forces in e.g. boundary layers.
4. SPH and turbulence modeling
5. Viscous forces and BCs (no-slip)
6. Other formulations for viscous terms.
7. Non-Newtonian fluids.

Further reading/studying

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